ON UNITARY UNIPOTENT REPRESENTATIONS OF p-ADIC GROUPS AND AFFINE HECKE ALGEBRAS WITH UNEQUAL PARAMETERS

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ABSTRACT. We determine the unitary dual of the geometric graded Hecke algebras with unequal parameters which appear in Lusztig's classification of unipotent representations for exceptional p-adic groups. The largest such algebra is of type F_4. Via the Barbasch-Moy correspondence of unitarity applied to this setting, this is equivalent to the identification of the corresponding unitary unipotent representations with real central character of the p-adic groups. In order for this correspondence to be applicable here, we show (following Lusztig's geometric classification, and Barbasch and Moy's original argument) that the set of tempered modules with real central character for a geometric graded Hecke algebra is linearly independent when restricted to the Weyl group.

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1. INTRODUCTION

1.1. The category of representations with unipotent cuspidal support for the rational forms inner to a split p-adic group of adjoint type is equivalent to a direct product of categories of affine Hecke algebras with unequal parameters ([Lu6]). We are interested in the problem of identifying the unitary representations in this category.

Let us briefly recall the Deligne-Langlands-Lusztig correspondence first. Let $\mathbb{F}$ denote a finite extension of the p-adic numbers, with residue field $\mathbb{F}_q$. Let $\mathcal{G}$ denote a connected adjoint quasi-simple algebraic group defined over the algebraic completion $\overline{\overline{\mathbb{F}}}$. Let Ψ_s be the inner class of $\mathcal{G}$ which contains the split $\mathbb{F}$-form of $\mathcal{G}$. For every $\mathcal{G}(\mathbb{F}) \in \Psi_s$, one defines the category of unipotent representations $\mathsf{Uni}(\mathcal{G}(\mathbb{F}))$ as follows. For every parahoric subgroup J of $\mathcal{G}(\mathbb{F})$, let $\overline{J}$ denote the reductive quotient (a reductive group over $\mathbb{F}_q$). For every unipotent cuspidal representation $\overline{\rho}$ of $\overline{J}$

(in the sense of Deligne-Lusztig), let ρ be the representation obtaining by pulling back to J. (The unipotent cuspidal representations appear very rarely.) Then let $\mathsf{Uni}_{(J,\rho)}(\mathcal{G}(\mathbb{F}))$ denote the category of admissible representations π of $\mathcal{G}(\mathbb{F})$ such that $\mathrm{Hom}_J[\pi : \rho]$ is nonzero and generates π. Set $\mathsf{Uni}(\mathcal{G}(\mathbb{F})) = \sqcup_{(J,\rho)}\mathsf{Uni}_{(J,\rho)}(\mathcal{G}(\mathbb{F}))$, and $\mathsf{Uni}(\mathcal{G}) = \sqcup_{\mathcal{G}(\mathbb{F})\in\Psi_s}\mathsf{Uni}(\mathcal{G}(\mathbb{F}))$. This is the "arithmetic side" of the correspondence.

For every (J, ρ) that appears, set $\mathcal{H}(\mathcal{G}(\mathbb{F}), J, \rho) = \mathrm{End}(\mathrm{Ind}_J^{\mathcal{G}(\mathbb{F})}(\rho))$. This is an affine Hecke algebra with unequal parameters. The prototype is the subcategory of representations generated by their vectors fixed under an Iwahori subgroup, that is, $\mathsf{Uni}(\mathcal{G}(\mathbb{F}), I, triv)$, where I is an Iwahori subgroup. The irreducible objects in this category are precisely the subquotients of the unramified principal series of $\mathcal{G}(\mathbb{F})$. By the fundamental results of Borel and Casselman, there is an equivalence of categories between $\mathsf{Uni}(\mathcal{G}(\mathbb{F}), I, triv)$ and $\mathcal{H}(\mathcal{G}(\mathbb{F}), I, triv)$-mod given by the functor $\pi \mapsto \pi^I := \mathrm{Hom}_I[\pi : triv]$. In general, by the results of Morris and Moy-Prasad, there is an equivalence of categories between $\mathsf{Uni}_{(J,\rho)}(\mathcal{G}(\mathbb{F}))$ and $\mathcal{H}(\mathcal{G}(\mathbb{F}), J, \rho)$-mod given by $\pi \mapsto \mathrm{Hom}_J[\pi : \rho]$.

On the "geometric side", one considers the complex simply-connected group G with root datum dual to $\mathcal{G}$. Kazhdan-Lusztig in the case of equal parameters, and Lusztig in general, gave a geometric construction of affine Hecke algebras, respectively affine graded Hecke algebras starting from G. Let L be a quasi-Levi subgroup of G (i.e, the centralizer of a semisimple element in G), with Lie algebra $\mathfrak{l}$, $\mathcal{C}$ a nilpotent class in $\mathfrak{l}$ which admits a cuspidal local system $\mathcal{L}$. (These are very rare.) Then one attaches an affine Hecke algebra $\mathcal{H}(G, L, \mathcal{C}, \mathcal{L})$ with unequal parameters, whose representation theory is classified in terms of perverse sheaves on certain graded nilpotent orbits of $\mathfrak{g}$. The truly remarkable realization of [Lu6] is that the two collections of affine Hecke algebras $\{\mathcal{H}(\mathcal{G}(\mathbb{F}), J, \rho) : G(\mathbb{F}) \in \Psi_s, (J, \rho)\}$ and $\{\mathcal{H}(G, L, \mathcal{C}, \mathcal{L}) : (L, \mathcal{C}, \mathcal{L})\}$ coincide, and via this identification, one translates the geometric classification to $\mathsf{Uni}(\mathcal{G})$.

Theorem ([Lu6]). *There is a one to one correspondence between* $\mathsf{Uni}(\mathcal{G})$ *and G-conjugacy classes in*

$$\{(s, e, \psi) : s \in G \ semisimple, e \in \mathfrak{g}, Ad(s)e = qe, \psi \in \widehat{A(s, e)}\}. \qquad (1.1.1)$$

Note that $e \in \mathfrak{g}$ is necessarily nilpotent. In (1.1.1), $A(s, e)$ denotes the group of components of the centralizer in G of s and e.

Remarks.
a) We emphasize that the left hand (arithmetic) side of the correspondence is a union over all rational forms of $\mathcal{G}$ inner to the split one.
b) An individual piece $\mathsf{Uni}(\mathcal{G}(\mathbb{F}))$, corresponding to a fixed rational form, is distinguished by the action of $\psi \in \widehat{A(s, e)}$ on the center $Z(G)$. For example, the split form $G(\mathbb{F})^s$ is parameterized by those ψ such that $\psi|_{Z(G)} = triv$. The Iwahori-spherical subcategory $\mathsf{Iw}(\mathcal{G}(\mathbb{F})^s)$ of the split form is parameterized by the representations ψ of Springer type, i.e., those which appear in the natural action of $A(s, e)$ on the Borel-Moore homology of the variety $\mathcal{B}_{s,e}$ of Borel subgroups of G containing both s and e ([KL]). The trivial $A(s, e)$-representation is of Springer type.
c) By [Re2] (also [BM4] in the Iwahori-spherical case), the representations in $\mathsf{Uni}(\mathcal{G}(\mathbb{F})^s)$ which are generic in the sense that they admit Whittaker models, are parameterized in theorem 1.1 by (s, e, ψ), where e is in the unique open orbit of

$Z_G(s)$ on $\mathfrak{g}_q := \{x \in \mathfrak{g} : Ad(s)x = qs\}$, and ψ is trivial. Here $Z_G(s)$ denotes the centralizer of s in G. In particular, these representations are all Iwahori-spherical.

d) The (K-)spherical representations of $\mathcal{G}(\mathbb{F})^s$ correspond to (s, e, ψ) with $e = 0$ and $\psi = triv$. They are Iwahori-spherical too. The Iwahori-Matsumoto involution interchanges spherical and generic representations and by [BM1, BM2], it preserves unitarity.

e) A first calculation we present (in section 4) as an example concerns unitary representations for $\mathcal{G} = E_6$. The dual G is also type E_6, simply connected and $Z(G) = \mathbb{Z}/3$. Let E_6^{nqs} denote any one of the two non-quasisplit form of type E_6 (denoted $^3E_6$ in the tables of [Ti]). On the arithmetic side, $\mathsf{Iw}(E_6^{nqs})$ is the category of Iwahori-spherical representations of E_6^{nqs}. On the geometric side, by [Lu5], there exist two cuspidal local systems for the Levi $L = 2A_2$ in E_6 (notation as in [Ca]), and the geometric Hecke algebras are isomorphic. The two cuspidal local systems (actually representations of the component group) are distinguished by how they act on $Z(G)$. By [Lu6], the geometric Hecke algebra and the Iwahori-Hecke are identical, of type G_2 with certain unequal parameters (see the tables in [Ti]). At the level of geometric parameters (1.1.1), the component group representations ψ which appear, act on $Z(G)$ like a fixed character of $\mathbb{Z}/3$ of order 3.

f) The hardest calculation in this paper refers to the case of unitary representations for $\mathcal{G} = E_7$. The dual G is also of type E_7, simply connected. Then $Z(G) = \mathbb{Z}/2$, and the two representations of $Z(G)$ correspond to the two rational forms in the split inner class. Let us denote by E_7^{ns} the nonsplit form. On the arithmetic side, $\mathsf{Iw}(E_7^{ns})$ is the category of Iwahori-spherical representations of E_7^{ns}. The corresponding Iwahori-Hecke algebra is of type F_4 with certain unequal parameters On the geometric side, there exists a cuspidal local system for the Levi $L = (3A_1)''$ in E_7. The affine Hecke algebra constructed from it has the same presentation with generators and relations as the Iwahori-Hecke one for E_7^{ns}. At the level of geometric parameters (1.1.1), the component group representations ψ which are nontrivial on $Z(G)$ appear. The nilpotent orbits of E_7 which allow such representations are in figure 2.

g) By changing short roots with long roots in an affine Hecke algebra which is not simply-laced, and scaling the unequal parameters accordingly, one obtains isomorphic Hecke algebras. Using this isomorphism, the same Hecke algebra as in e) controls two subcategories of unipotent representations for $\mathcal{G}(\mathbb{F}) = E_8^s$ (split) and a parahoric of type E_6, while the Hecke algebra in f) controls a subcategory for $\mathcal{G}(\mathbb{F}) = E_8^s$ and a parahoric of type D_4. The weight structure of the discrete series modules for these latter Hecke algebras, and their formal degrees, were calculated in [Re1]. We verified that the dimensions of the discrete series, which we obtained by finding their W-structure, agree with the dimensions listed in [Re1]. Note also that by the Barbasch-Moy transfer of unitarity (see section 1.2) one can obtain interesting correspondences between unitary representations whenever two categories of unipotent representations are controlled by the same affine Hecke algebra.

1.2.   The Iwahori-Hecke algebra is a convolution algebra of complex-valued functions, and therefore has a natural $*$-operation: $f \mapsto f^*$, $f^*(g) := \overline{f(g^{-1})}$. One defines hermitian and unitary modules with respect to this $*$-operation. The approach to the determination of Iwahori-spherical unitary representations of a split p-adic group via a reduction to Iwahori-Hecke algebra modules with real central

("infinitesimal") character was introduced in [BM1, BM2]. It is based on the geometric classification, and the use of the graded Hecke algebra of [Lu1].

Let us recall succinctly the machinery of Barbasch-Moy's preservation of unitarity under the real central character assumption. The central characters of $\mathcal{H}(G, I, triv)$ are in one to one correspondence with G-conjugacy classes of semisimple elements s in G. This is the semisimple element s appearing in the correspondence (1.1.1). We say that a central character s is real if the elliptic part of s in the polar decomposition is central in G. The element s is sometimes referred to as the infinitesimal character of the representation.

Theorem ([BM1]). *In the equivalence of categories $\pi \mapsto \pi^I$ between $\mathsf{lw}(\mathcal{G}(\mathbb{F})^s)$ and $\mathcal{H}(G, I, triv)$-mod, hermitian and unitary modules with real infinitesimal character correspond, respectively.*

One of the basic ingredients for the proof is the signature character of a representation introduced in [Vo1]. If π a hermitian representation in $\mathsf{lw}(\mathcal{G}(\mathbb{F})^s)$, let $\Sigma(\pi)$ denote its signature character, and $\theta_K(\pi)$ its K-character (see [BM1]). Similarly, if $\overline{\pi}$ is a module in $\mathcal{H}(G, I, triv)$-mod, let $\Sigma(\overline{\pi})$ denote its signature character, and $\theta_W(\overline{\pi})$ its W-character. (Recall that $\mathcal{H}(G, I, triv)$ has the subalgebra of functions supported on K, which is isomorphic with the Weyl group W of G.) The proof that π^I unitary implies π unitary (the other implications are straightforward) in [BM1] has the following steps:

(1) There exist tempered representations $\pi_1, \ldots, \pi_\ell$ in $\mathsf{lw}(\mathcal{G}(\mathbb{F})^s)$ such that

$$\Sigma(\pi) = \left(\sum_i a_i \theta_K(\pi_i), \sum_i b_i \theta_K(\pi_i)\right), \tag{1.2.1}$$

for some integers a_i, b_i. (This is proved by induction on the length of the Langlands parameter, and uses the fact that $\mathsf{lw}(\mathcal{G}(\mathbb{F})^s)$ is closed under taking subquotients.)

(2) A module π' in $\mathsf{lw}(\mathcal{G}(\mathbb{F})^s)$ is tempered if and only if $(\pi')^I$ is tempered in $\mathcal{H}(G, I, triv)$-mod. (One defines temperedness via Casselman's criterion, so the notion makes sense in both settings.)

(3)

$$\Sigma_W(\pi^I) = \left(\sum_i a_i \theta_W(\pi_i^I), \sum_i b_i \theta_W(\pi_i^I)\right), \tag{1.2.2}$$

for the same a_i, b_i as in (1.2.1). (One needs here not only that the irreducibles correspond in the equivalence of categories, but also the fact that the standard modules correspond.)

(4) The set $\{\theta_W(\overline{\pi}) : \overline{\pi}$ is tempered with real central character$\}$ is linearly independent in the Grothendieck group of W, in fact it gives a basis. (This proof is based on the geometric realization of the W-structure of tempered modules and the results of Borho-MacPherson.)

(5) Then one sees that if π^I is unitary, $\Sigma(\pi^I)$ being positive definite, it implies by (4) that $b_i = 0$, for all i, and so $\Sigma(\pi)$ is positive definite as well.

The essential step in this correspondence is (4). As stated, it does not hold true if one drops the "real central character" assumption. Even if one considers the weaker (but sufficient) case of tempered modules whose central character have a fixed elliptic part s_e, this statement is still false in general. The subject of [BM2] is to show that, however, one still has that, for the tempered modules (with a fixed

elliptic part s_e) which appear in the equation (1.2.2), $\sum_i b_i \theta_W(\pi_i^I) = 0$ implies $b_i = 0$. For this one uses (a variant of) the reduction to the graded Hecke algebra ([Lu1]).

In this setting, the spherical (equivalently, the generic Iwahori-spherical) unitary dual for all split p-adic groups is determined: for classical types in [BM3] and [Ba], for types G_2, F_4 in [Ci1], and for types E in [BC1]. By different methods, the complete unitary dual for types $GL(n)$ and G_2 had been previously obtained in [Ta] and [Mu], respectively.

For non-generic modules (of types other than A or G_2) less is known: the unitary dual with real central character for the graded Hecke algebra with equal parameters is known for F_4 ([Ci1]) and E_6 ([Ci2]). It is not known for classical groups, except in some small rank cases.

When one attempts to extend theorem 1.2 to the categories $\mathsf{Uni}(\mathcal{G}(\mathbb{F}), J, \rho)$, the first thing to prove is item (4) above: the linear independence of W-characters for tempered modules with real central character in the affine Hecke algebra with unequal parameters appearing on the geometric side. This is discussed in section 3 of this paper. We work, as we may since the central character is assumed real, in the equivalent setting of the graded Hecke algebra.

1.3. In [BC1], one proposed a direct method of matching signatures of hermitian forms for irreducible modules of a graded Hecke algebra with equal parameters $\mathbb{H}$ with signatures of hermitian forms on spherical modules of smaller graded Hecke algebras $\mathbb{H}(\mathcal{O})$ constructed from root systems of centralizers of nilpotent complex orbits $\mathcal{O}$. This method proved particularly effective in the determination of the unitary generic modules, but it can be applied for the signatures of non-generic modules as well.

In this paper, we apply this idea to the setting of geometric Hecke algebras $\mathbb{H}$ with unequal parameters ([Lu2, Lu3, Lu4]). As an application, we determine the unitary dual of certain Hecke algebras of types G_2 and F_4, with unequal parameters, which appear in [Lu6], in the classification of unipotent representations of p-adic groups.

We present next the philosophy of matching signatures and unitary duals. The geometric classification of $\mathbb{H}$ is controlled by a complex group G, with Lie algebra $\mathfrak{g}$, and Cartan subalgebra $\mathfrak{t}$, a Levi subgroup L with Lie algebra $\mathfrak{l}$, and a cuspidal local system $\mathcal{L}$ supported on some nilpotent orbit $\mathcal{C}$ of $\mathfrak{l}$ (see section 3). Note that the rank of G may be larger than that of $\mathbb{H}$, e.g., the most interesting example in this paper is an algebra of type F_4 parameterized by G simply connected of type E_7. Let V be a simple $\mathbb{H}$-module. The graded Hecke algebra version of the geometric classification (1.1.1) says that V is parameterized by a triple $(s, \mathcal{O}_e, \psi)$, where s is a (conjugacy class of a) semisimple element in $\mathfrak{t}$, $\mathcal{O}_e$ is a $Z_G(s)$-orbit on $\mathfrak{g}_1 = \{x \in \mathfrak{g} : [s, x] = x\}$, and ψ is a certain representation of the component group $A(s, e)$. The orbit $\mathcal{C}$ is necessarily distinguished in $\mathfrak{l}$ (in the sense of Bala-Carter), and there are restrictions about which ψ can appear depending on the cuspidal local system $\mathcal{L}$. The nilpotent representative e can be completed to a Jacobson-Morozov triple $\{e, h, f\}$ of $\mathcal{O} = G \cdot \mathcal{O}_e$, such that $h \in \mathfrak{t}$. Let $Z(\mathcal{O})$ denote the centralizer in G of $\{e, h, f\}$. It is a reductive group, possibly disconnected. Let $A(\mathcal{O})$ denote its group of components, and $\mathfrak{z}(\mathcal{O})$ its Lie algebra. Then s can be written as

$$s = \frac{1}{2}h + \nu, \tag{1.3.1}$$

where ν is a semisimple element of $\mathfrak{z}(\mathcal{O})$.

Now fix a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, together with a representation ϕ of the component group $A(\mathcal{O})$. The requirement is that the pair $(\mathcal{O}, \phi)$ appears in the generalized Springer correspondence ([Lu5]), and so it has a representation μ_0 of the Weyl group W defining $\mathbb{H}$ attached to it. This is not the Weyl group of G, but rather $W = N_G(L)/L$. (The fact that this is a Coxeter group is due to the very particular form L must have in order to allow a cuspidal local system.) Let us denote by $\mathcal{U}(\mathcal{O}, \phi)$ the set of unitary $\mathbb{H}$-modules with *lowest W-type* μ_0, in the sense of sections 3.4-3.6. The triple $(s, \mathcal{O}_e, \psi)$ parameterizing a module V with lowest W-type μ_0 must satisfy $\mathcal{O}_e \subset \mathcal{O}$, and $\mathrm{Hom}_{A(s,e)}[\phi : \psi] \neq 0$.

We attach a Hecke algebra $\mathbb{H}(\mathfrak{z}(\mathcal{O}), c(\phi))$ with the root system that of $\mathfrak{z}(\mathcal{O})$, and certain parameters $c(\phi)$ depending on ϕ, and consider $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}), c(\phi)))$, the set of unitary spherical $\mathbb{H}(\mathfrak{z}(\mathcal{O}), c(\phi))$-modules. By the equation (1.3.1), $s \mapsto \nu$, to a module $V = V_{s, \mathcal{O}_e, \psi}$ with lowest W-type μ_0, we may attach a spherical module $\rho(V)$ of $\mathbb{H}(\mathfrak{z}(\mathcal{O}), c(\phi))$ parameterized by ν (*i.e.*, the spherical subquotient of the minimal principal series defined by ν).

The hope would be that, in this way, the set $\mathcal{U}(\mathcal{O}, \phi)$ can be put in a correspondence with $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}), c(\phi)))$. This is motivated by the case of generic unitary modules of Hecke algebras with equal parameters ([Ba, BC1, Ci1]), where a particular case of this correspondence worked almost perfectly. More precisely, that was the case of $\phi = triv$, so that $\mathbb{H}$ is a Hecke algebra with equal parameters of the same type as G (L is a maximal torus, and $\mathcal{C} = 0$). The attached Hecke algebra is $\mathbb{H}(\mathfrak{z}(\mathcal{O}), 1)$, meaning that it has equal parameters too. Let us denote by $\mathcal{GU}(\mathcal{O}, triv)$ and $\mathcal{GSU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}), 1)$ the corresponding subsets consisting of (unitary) generic representations, in the sense of containing the sign W-type. It was shown in the above references that, except for a handful of cases (none in the classical types, in G_2 or E_6, one in F_4 and in E_7, and six in E_8), the sets $\mathcal{GU}(\mathcal{O}, triv)$ and $\mathcal{GSU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}), 1)$ are in one-to-one correspondence. In more generality, such a relation is not as perfect, but it still gives serious information about $\mathcal{U}(\mathcal{O}, \phi)$.

There are also two immediate technical problems. One obvious problem is that, in general, an irreducible module may have more than one lowest W-type, so it is actually better to match $\mathcal{U}(\mathcal{O}, \phi)$ with a "quasi-spherical" unitary dual of an extension (by outer automorphisms) of the Hecke algebra defined from $\mathfrak{z}(\mathcal{O})$. A second issue is that, unlike the case of representations of real reductive groups, a lowest W-type may appear with multiplicity greater than one. But if this is the case, it is a fact (only empirical to our knowledge) that there exists another lowest W-type with multiplicity one. This fact is needed in order to be able to define a normalization of the intertwining operators involved in the calculation of signatures of hermitian forms. However, this second difficulty only arises for geometric algebras with equal parameters. We ignore these problems for now and refer to [BC1] and [Ci2] for details and examples of these phenomena.

In order to exhibit a relation between $\mathcal{U}(\mathcal{O}, \phi)$ and $\mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}), \phi))$, for every $(\mathcal{O}, \phi)$, we find a set of W-types $\{\mu_0, \mu_1, \ldots, \mu_l\}$, and a corresponding set of $W(\mathfrak{z}(\mathcal{O}))$-representations $\{\rho(\mu_0), \rho(\mu_1), \ldots, \rho(\mu_l)\}$,

$$\mu_j \longleftrightarrow \rho(\mu_j), \qquad (1.3.2)$$

such that:

(1) if V is hermitian but $\rho(V)$ is not, then the hermitian form on V is indefinite, *i.e.*, V is not unitary;

(2) if V is hermitian and $\rho(V)$ is hermitian, then the signature of the hermitian form of V restricted to the W-type μ_j is identical with the signature of the hermitian form of $\rho(V)$ restricted to the $W(\mathfrak{z}(\mathcal{O}))$-representation $\rho(\mu_j)$, for all $0 \leq j \leq l$.

We explain the relation between μ_j and $\rho(\mu_j)$ (see [BC1]). Let $\mathfrak{m} \supset \mathfrak{l}$ denote a Levi subalgebra of $\mathfrak{g}$, which is Bala-Carter for $\{e, h, f\}$, and let $M \subset G$ be the corresponding Levi subgroup. So e is distinguished in $\mathfrak{m}$. There exists a geometric Hecke subalgebra $\mathbb{H}_M$ of $\mathbb{H}$ which is parameterized by $(L, \mathcal{L})$ in M. By [Lu4], the irreducible $\mathbb{H}_M$-modules parameterized by (h, e, ψ'), for any allowable $\psi' \in \widehat{A_M(s, e)}$, are discrete series. We find a ψ' such that via the natural map $A_M(h, e) = A_M(s, e) \to A(s, e)$, $\mathrm{Hom}_{A_M}(h, e)[\psi, \psi'] \neq 0$, and realize $V_{s,e,\psi}$ as a summand of the Langlands subquotient of the standard induced module $X(M, \sigma, \nu) = \mathbb{H} \otimes_{\mathbb{H}_M} (\sigma \otimes \mathbb{C}_\nu)$. Then $\mathrm{Hom}_W[\mu_j : X(M, \sigma, \nu)]$ carries a natural action of $W(\mathfrak{z}(\mathcal{O}))$ (independent of ν) and we call the resulting representation $\rho(\mu_j)$.

Of course, this procedure is efficient only if we find a sufficiently large number of matching types in this way. This is our main criterion for proving non-unitarity of $\mathbb{H}$-modules. Ideally, if sufficiently many W-types are matched, one would be able to conclude that a hermitian module V is unitary only if $\rho(V)$ is unitary. This doesn't always happen, and in fact we find examples here of $\mathcal{U}(\mathcal{O}, \phi)$'s both smaller or larger than expected. We exemplify this at the end of the introduction.

There is also the case of spherical unitary $\mathbb{H}$-modules, whose lowest W-type is the trivial. In that case, $\mathcal{O} = G \cdot \mathcal{C}$ (it is the minimal orbit appearing in the parameterization), $\mathfrak{z}(\mathcal{O})$ has the same type as $\mathbb{H}$, and the above correspondence is a tautology. We determine the spherical unitary $\mathbb{H}$-modules by other methods, known to the experts. In particular, we use the Iwahori-Matsumoto involution IM (definition (4.2.1)), which preserves unitarity. If V is a spherical $\mathbb{H}$-module, then $IM(V)$ contains the sign W-type. If $IM(V) \neq V$, then $IM(V)$ is parameterized by a larger orbit, for which the unitary set has already been determined. But in order to apply this method, we need to determine explicitly which unitary irreducible modules have the sign W-type for any given pair $(\mathcal{O}', \phi')$. This is achieved by computing composition series as part of section 5.3, and the result is in table 4.

So this approach leaves the case $V = IM(V)$ and V spherical, equivalently, V must be an irreducible spherical principal series. The unitarity of such representations is discussed in section 5.5, and the methods are again well-known: continuous deformations of parameters and unitary induction, but also certain explicit calculation of signatures.

1.4. We give an outline of the paper. In section 2.1-2.7, we present the necessary definitions and background on the representation theory of affine and graded Hecke algebras, assuming the parameters are arbitrary. We will mostly work with the affine graded Hecke algebra. In particular, we recall the Langlands classification (actually, the reduction to tempered modules) as in [Ev], the unitarity of tempered modules, as it follows from [Op1], and the results about the Hermitian forms and intertwining operators from [BM3]. In section 2.8, we present a type of reduction to *real* central ("infinitesimal") character for the unitary dual of the Hecke algebra,

which is the complete analogue of the result for real reductive groups, as in [Kn], XVI.4. This is well-known to the specialists.

From section 3 on, we restrict to the case of geometric Hecke algebras. We recall the relevant results about the classification of simple $\mathbb{H}$-modules, and the generalized Springer correspondence ([Lu5]). An important consequence is that the set of irreducible tempered $\mathbb{H}$-modules with real central character are linearly independent in the Grothendieck group of W. Therefore the method of [BM1] holds, and the correspondence with unitary (unipotent) representations of p-adic groups can be established. So for geometric Hecke algebras, the unitarity of tempered $\mathbb{H}$-modules could also be obtained from the correspondence with the p-adic group.

In section 4, we analyze Hermitian forms and intertwining operators for simple modules. We restrict, as we may, to modules with real central character. As an easy application, we present the spherical unitary dual for type G_2 with arbitrary unequal parameters. (In fact, if we factor in the Iwahori-Matsumoto involution, and the tempered modules, one obtains all the unitary modules with real central character of G_2.)

Section 5 presents a more interesting application. We determine the unitary dual of the Hecke algebra (of type F_4) constructed in [Lu2] from a cuspidal local system on the principal nilpotent orbit in the Levi $(3A_1)''$ of the simply connected E_7. Besides the equal parameter case, this is, essentially, the only Hecke algebra of type F_4 which appears in the classification of unipotent representations of p-adic groups from [Lu6] (the equal parameter $\mathbb{H}(F_4)$ was treated in [Ci1]). We also remark that the Hecke algebras of types B/C with unequal parameters which appear in the classification of the unipotent exceptional p-adic groups are parabolic subalgebras of this one.

The main results are theorem 5.1, proposition and corollary 5.5. Although the methods employed are mostly uniform, details need to be checked case by case for each nilpotent orbit in E_7 which appears in the classification (see figure 2). As mentioned above, we use the fact that IM preserves unitarity of modules. In order to use IM, we need to compute the decompositions of standard modules and the W-structure of unitary modules.

As a consequence of the calculations with intertwining operators in section 5, we obtain the W-structure of standard modules for the cuspidal local system $(3A_1)''$ in E_7 (section 5.6). This is also (at least in principle) computable by the generalized Green polynomials algorithms in [Lu7]. An implementation of such algorithms for small rank classical groups was realized in [LS].

1.5. **Examples.** We conclude the introduction with two examples relative to the matching of signatures and unitary duals for this unequal parameter $\mathbb{H}$ of type F_4. The nilpotent orbits are of type E_7.

(a) $\mathcal{O} = A_2 + 3A_1$, $A(\mathcal{O}) = \mathbb{Z}/2\mathbb{Z}$, $\mathfrak{z}(\mathcal{O}) = G_2$, $\phi = sgn$. The intertwining operator calculations indicate a matching with the spherical unitary dual of a Hecke algebra $\mathbb{H}(G_2, (2, 1))$: type G_2 with parameter 2 on the long root and 1 on the short root. $W(G_2)$ has six representations, in the notation of [Ca]: $(1, 0)$, $(1, 3)'$, $(1, 3)''$, $(1, 6)$, $(2, 1)$, $(2, 2)$. There are two minimal subsets of $\widehat{W(G_2)}$ which determine $\mathcal{SU}(\mathbb{H}(G_2, (2, 1)))$, in the sense that a hermitian module is unitary if and only if the hermitian form is positive definite on this set of types:

$\{(1,0),(2,1),(2,2)\}$ or $\{(1,0),(1,3)',(1,3)'',(2,1)\}$. It turns out there are $W(F_4)$-types μ_j such that one realizes as $\rho(\mu_j)$ (see the explanation around equation (1.3.2)) the $W(G_2)$-representations $\{(1,0),(1,3)',(2,1),(1,6)\}$. So we cannot conclude that $\mathcal{U}(A_2 + 3A_1, sgn)$ is a subset of $\mathcal{SU}(\mathbb{H}(G_2,(2,1)))$, and in fact, we find that $\mathcal{U}(A_2 + 3A_1, sgn) \supsetneq \mathcal{SU}(\mathbb{H}(G_2,(2,1)))$ (see table 3 and figure 3).

(b) $\mathcal{O} = D_4(a_1) + A_1$, $A(\mathcal{O}) = (\mathbb{Z}/2\mathbb{Z})^2$, $\mathfrak{z}(\mathcal{O}) = 2A_1$. There are two representations ϕ_1 and ϕ_2 of $A(\mathcal{O})$ which enter the generalized Springer correspondence with $\widehat{W(F_4)}$ ([Spa1]): ϕ_1 corresponds to a nine-dimensional $W(F_4)$-representation, and ϕ_2 to a two-dimensional. We consider here ϕ_1. The calculations indicate that the matchings should be with the spherical unitary dual of $\mathbb{H}(2A_1,(5,5))$: type $A_1 + A_1$, and the parameter is 5 on both simple roots. The set $\mathcal{SU}(\mathbb{H}(2A_1,(5,5)))$ is detected on $\{triv \otimes triv, triv \otimes sgn, sgn \otimes triv\} \subset \widehat{W(2A_1)}$. All three representations are matched by certain μ_j's, so we can conclude that $\mathcal{U}(D_4(a_1) + A_1, \phi_1) \subset \mathcal{SU}(\mathbb{H}(2A_1,(5,5)))$. The problem here is that there is extra reducibility in the standard modules for $\mathbb{H}$ parameterized by $(\mathcal{O}, \phi)$ which is not detected by the hermitian form on the $W(F_4)$-types μ_j. What we find is that $\mathcal{U}(D_4(a_1) + A_1, \phi_1) \subsetneq \mathcal{SU}(\mathbb{H}(2A_1,(5,5)))$ (see table 3).

2. Preliminaries

2.1. Let $(X, \check{X}, R, \check{R}, \Pi)$ be a fixed based root datum. Denote by R^+ the positive roots determined by Π, and by $\check{R}^+$ the corresponding positive coroots. Define $\mathfrak{h} = \check{X} \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathfrak{h}^* = X \otimes_{\mathbb{Z}} \mathbb{C}$. Let W be the finite Weyl group generated by the set S of simple reflections in the roots of Π. Denote by $Q = \mathbb{Z}R \subset X$ the root lattice, and set $\Omega = X/Q$.

Let $\widetilde{W} = W \ltimes X$ be the extended Weyl group, and $W_{\text{aff}} = W \ltimes Q$ be the affine Weyl group with the set of simple affine reflections S_{aff}. Recall that W_{aff} is normal in $\widetilde{W}$, and $\widetilde{W}/W_{\text{aff}} \cong \Omega$. Let $\ell : \widetilde{W} \to \mathbb{Z}_{\geq 0}$ denote the length function ([Lu1]): it is the extension of the (Coxeter) length function of W_{aff} and it is identically 0 on Ω.

2.2. Let $c : R \to \mathbb{Z}_{>0}$ be a function such that $c_\alpha = c_\beta$, whenever α and β are W-conjugate. Let r denote an indeterminate. As a vector space,

$$\mathbb{H} = \mathbb{C}[W] \otimes \mathbb{C}[r] \otimes \mathbb{A}, \tag{2.2.1}$$

where $\mathbb{A}$ is the symmetric algebra over $\mathfrak{h}^*$. The generators are $t_w \in \mathbb{C}[W]$, $w \in W$ and $\omega \in \mathfrak{h}^*$. The relations between the generators are:

$$\begin{aligned}
&t_w t'_w = t_{ww'}, &&\text{for all } w, w' \in W; \\
&t_s^2 = 1, &&\text{for any simple reflection } s \in W; \\
&\omega t_s = t_s s(\omega) + r c_\alpha \omega(\check{\alpha}), &&\text{for simple reflections } s = s_\alpha \in S.
\end{aligned}$$
$$\tag{2.2.2}$$

Assume the root system R is irreducible. If R is simply-laced, the function c must be constant. In the two root lengths case, we will denote the function c by the pair $(c_l, c_s) \in \mathbb{Z}_{>0}^2$, where c_l, c_s specify the values of c on the long and short roots, respectively (in this order). Let $k \in \mathbb{Z}_{>0}$ be the ratio $k = (\alpha_l, \alpha_l)/(\alpha_s, \alpha_s)$ for some long root α_l and short root α_s. Then it is straightforward to see that $\mathbb{H}(R, (c_1, c_2)) \cong \mathbb{H}(R, (kc_2, c_1))$. For example, $\mathbb{H}(F_4, (1, 2)) \cong \mathbb{H}(F_4, (4, 1))$.

2.3.   By [Lu1], the center of $\mathbb{H}$ is $\mathbb{C}[r] \otimes \mathbb{A}^W$. On any simple (finite dimensional) $\mathbb{H}$-module, the center of $\mathbb{H}$ acts by a character, which we will call a *central character*. The central characters correspond to W-conjugacy classes of semisimple elements $(r_0, s) \in \mathbb{C} \oplus \mathfrak{h}$.

   We decompose $\mathfrak{h}$ into real and imaginary parts; the complex conjugation $\bar{}$ of $\mathbb{C}$ induces a conjugation on $\mathfrak{h}$. We set $\mathfrak{h}_{\mathbb{R}} = \{a \in \mathfrak{h} : \bar{a} = a\}$ and $\mathfrak{h}_{i\mathbb{R}} = \{a \in \mathfrak{h} : \bar{a} = -a\}$. For every $s \in \mathfrak{h}$, we have then a unique decomposition $s = Re(s) + \sqrt{-1}Im(s)$, where $Re(s) \in \mathfrak{h}_{\mathbb{R}}$ and $Im(s) \in \mathfrak{h}_{i\mathbb{R}}$.

Definition. *A central character (r_0, s) is called real if $(r_0, s) \in \mathbb{R} \oplus \mathfrak{h}_{\mathbb{R}}$.*

2.4. $\mathbb{H}$ has a $*$- operation given on generators as follows (as in [BM2], section 5):

$$t_w^* = t_{w^{-1}}, \ w \in W; \ r^* = r; \tag{2.4.1}$$

$$\omega^* = -\overline{\omega} + r \sum_{\alpha \in R^+} c_\alpha \overline{\omega}(\check{\alpha}) t_{s_\alpha}, \ \omega \in \mathfrak{h}^*.$$

   The $\mathbb{H}$-module V is called *Hermitian* if it admits a Hermitian form $\langle \ , \ \rangle$ such that

$$\langle x \cdot v_1, v_2 \rangle = \langle v_1, x^* \cdot v_2 \rangle, \ \text{for all } v_1, v_2 \in V, \ x \in \mathbb{H}. \tag{2.4.2}$$

It is called *unitary*, if in addition the Hermitian form is positive definite.

2.5.   We present the Langlands classification for $\mathbb{H}$ as in [Ev].

   If V is a (finite dimensional) simple $\mathbb{H}$-module, $\mathbb{A}$ induces a generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}} V_\lambda. \tag{2.5.1}$$

Call λ a *weight* if $V_\lambda \neq 0$.

Definition. *The irreducible module σ is called tempered if $\omega_i(\lambda) \leq 0$, for all weights $\lambda \in \mathfrak{h}$ of σ and all fundamental weights $\omega_i \in \mathfrak{h}^*$. If σ is tempered and $\omega_i(\lambda) < 0$, for all λ, ω_i as above, σ is called a discrete series.*

   For every $\Pi_M \subset \Pi$, define $R_M \subset R$ to be the set of roots generated by Π_M, $\check{R}_M \subset \check{R}$ the corresponding set of coroots, and $W(M) \subset W$ the corresponding Weyl subgroup. Let $\mathbb{H}_M$ be the Hecke algebra attached to $(\mathfrak{h}, R_M)$. It can be regarded naturally as a subalgebra of $\mathbb{H}$.

   Define $\mathfrak{t} = \{\nu \in \mathfrak{h} : \langle \alpha, \nu \rangle = 0, \text{ for all } \alpha \in \Pi_M\}$ and $\mathfrak{t}^* = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \check{\alpha} \rangle = 0, \text{ for all } \alpha \in \Pi_M\}$. Then $\mathbb{H}_M$ decomposes as

$$\mathbb{H}_M = \mathbb{H}_{M_0} \otimes S(\mathfrak{t}^*),$$

where $\mathbb{H}_{M_0}$ is the Hecke algebra attached to $(\mathbb{C}\langle \Pi_M \rangle, R_M)$.

   We will denote by $I(M, U)$ the induced module $I(M, U) = \mathbb{H} \otimes_{\mathbb{H}_M} U$.

Theorem ([Ev])**.**

(1) *Every irreducible $\mathbb{H}$-module is a quotient of a standard induced module $X(M, \sigma, \nu) = I(M, \sigma \otimes \mathbb{C}_\nu)$, where σ is a tempered module for $\mathbb{H}_{M_0}$, and $\nu \in \mathfrak{t}^+ = \{\nu \in \mathfrak{t} : Re(\alpha(\nu)) > 0, \text{ for all } \alpha \in \Pi \setminus \Pi_M\}$.*
(2) *Assume the notation from (1). Then $X(M, \sigma, \nu)$ has a unique irreducible quotient, denoted by $L(M, \sigma, \nu)$.*
(3) *If $L(M, \sigma, \nu) \cong L(M', \sigma', \nu')$, then $M = M'$, $\sigma \cong \sigma'$ as $\mathbb{H}_{M_0}$-modules, and $\nu = \nu'$.*

We will call a triple (M, σ, ν) as in theorem 2.5 a *Langlands parameter*.

2.6.   We need to recall some results about the affine (Iwahori-)Hecke algebra as in [Op1], and their implications for $\mathbb{H}$. Fix $q > 1$. The *affine Hecke algebra* $\mathcal{H}$ is the associative $\mathbb{C}$-algebra with basis $\{T_w : w \in \widetilde{W}\}$ and set of parameters $d : \Pi_{\mathrm{aff}} \to \mathbb{Z}_{>0}$ which satisfies the relations:

(1) $T_w T_{w'} = T_{ww'}$ if $w, w' \in \widetilde{W}$, with $\ell(ww') = \ell(w) + \ell(w')$;
(2) $(T_{s_\alpha} + 1)(T_{s_\alpha} - q^{d_\alpha}) = 0$, for all simple affine reflections $s_\alpha \in S_{\mathrm{aff}}$.

This is the Iwahori-Matsumoto presentation of $\mathcal{H}$.

The $*$-operation for $\mathcal{H}$ is given by $T_w^* = T_{w^{-1}}$. The algebra $\mathcal{H}$ has a *trace*, i.e., a linear functional $\tau : \mathcal{H} \to \mathbb{C}$, given by

$$\tau(T_w) = \begin{cases} 1, & w = 1 \\ 0, & w \neq 1 \end{cases}, \quad w \in \widetilde{W}. \tag{2.6.1}$$

One checks that τ defines an inner product on $\mathcal{H}$ via

$$(x, y) := \tau(x^* y), \; x, y \in \mathcal{H}, \tag{2.6.2}$$

and the basis $\{T_w : w \in \widetilde{W}\}$ is orthogonal with respect to $(\ ,\ )$.

Definition. *Let $\mathfrak{H}$ be the Hilbert space completion of $\mathcal{H}$ with respect to $(\ ,\ )$.*

2.7.   The algebra $\mathcal{H}$ admits a second presentation due to Bernstein and Lusztig. The generators are $\{T_w : w \in W\}$ and $\{\theta_x : x \in X\}$ with relations:

(1) $T_w T_{w'} = T_{ww'}$ if $w, w' \in W$, with $\ell(ww') = \ell(w) + \ell(w')$;
(2) $(T_{s_\alpha} + 1)(T_{s_\alpha} - q^{d_\alpha}) = 0$, for all simple reflections $s_\alpha \in S$;
(3) $\theta_x \theta'_x = \theta_{x+x'}$, $x, x' \in X$, and $\theta_0 = 1$;
(4) $\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} = (q^{d_\alpha} - 1)\frac{\theta_x - \theta_{s_\alpha(x)}}{1 - \theta_{-\alpha}}$, $x \in X, s_\alpha \in S$.

Denote by $\mathcal{H}_W$ and $\mathcal{A}$ the subalgebras generated by $\{T_w\}$ and $\{\theta_x\}$ respectively. We refer the reader to [Lu1] for the more general version of this definition of $\mathcal{H}$ and for the relation between the two presentations.

Note that the trace τ is easy to define using the Iwahori-Matsumoto presentation, but not so if one uses instead the Bernstein-Lusztig presentation. (See [Op2] for details.) Similarly to section 2.5, one defines *tempered* and *discrete series* modules for $\mathcal{H}$ using the weights of the abelian subalgebra $\mathcal{A}$ ("Casselman's criteria", see [Lu1] and [Op1]).

Proposition ([Op1],2.22)**.** *A finite dimensional representation σ of $\mathcal{H}$ is a discrete series if and only if it is a subrepresentation of $\mathfrak{H}$. In particular, σ is a unitary module of $\mathcal{H}$.*

A standard argument then implies the unitarity of tempered $\mathcal{H}$-modules as well. Moreover, we can transfer this result to the graded Hecke algebra, using the correspondence between tempered modules of $\mathcal{H}$ and tempered modules of $\mathbb{H}$ (theorem 9.3 in [Lu1]) and the preservation of unitarity (theorem 4.3 in [BM2]).

2.8. Given a module V, let V^h denote the Hermitian dual. Every element $x \in \mathbb{H}$ can be written uniquely as $x = \sum_{w \in W/W(M)} t_w x_w$, with $x_w \in \mathbb{H}_M$. Let $\epsilon_M : \mathbb{H} \to \mathbb{H}_M$ be the map defined by $\epsilon_M(x) = x_1$, that is, the component of the identity element $1 \in W$. In the particular case $\Pi_M = \emptyset$, we will denote the map by $\epsilon : \mathbb{H} \to \mathbb{A}$.

Lemma ([BM3], 1.4). *If U is module for $\mathbb{H}_M$, and $\langle \ , \ \rangle_M$ denotes the Hermitian pairing with U^h, then the Hermitian dual of $I(M, U)$ is $I(M, U^h)$, and the Hermitian pairing is given by*

$$\langle t_x \otimes v_x, t_y \otimes v_y \rangle_h = \langle \epsilon_M(t_y^* t_x) v_x, v_y \rangle_M, \ x, y \in W/W(M), \ v_x, v_y \in U.$$

Applying this result to a Langlands parameter (M, σ, ν) as in section 2.5, we find that the Hermitian dual of $X(M, \sigma, \nu)$ is $I(M, \sigma \otimes \mathbb{C}_{-\overline{\nu}})$.

Let w_0 denote the longest Weyl group element in W, and let $W(w_0 M)$ be the subgroup of W generated by the reflections in $w_0 R_M$. Let w_m denote a shortest element in the double coset $W(w_0 M) w_0 W(M)$. Then $w_m \Pi_M$ is a subset of Π, which we denote by $\Pi_{w_m M}$.

Proposition ([BM3], 1.5). *The Hermitian dual of the irreducible Langlands quotient $L(M, \sigma, \nu)$ is $L(w_m M, w_m \sigma, -w_m \overline{\nu})$. In particular, $L(M, \sigma, \nu)$ is Hermitian if and only if there exists $w \in W$ such that*

$$wM = M, \quad w\sigma \cong \sigma \quad and \quad w\nu = -\overline{\nu}.$$

If this is the case, we will denote by $a_w : w\sigma \to \sigma$ the corresponding isomorphism.

Let $w = s_1 \dots s_k$ be a reduced decomposition of w. For each simple root α, define

$$r_{s_\alpha} = t_{s_\alpha} \alpha - c_\alpha; \quad r_w = r_{s_{\alpha_1}} \dots r_{s_{\alpha_k}}. \tag{2.8.1}$$

Lemma 1.6 in [BM3] (based on proposition 5.2 in [Lu1]) proves that r_w does not depend on the reduced expression of w.

Assume $L(M, \sigma, \nu)$ is Hermitian. Define

$$\mathcal{A}(M, \sigma, \nu) : X(M, \sigma, \nu) \to I(M, \sigma \otimes \mathbb{C}_{-\overline{\nu}}), \ x \otimes (v \otimes 1_\nu) \mapsto x r_w \otimes (a_w(v) \otimes 1_{-\overline{\nu}}). \tag{2.8.2}$$

One can verify, as in section 1.6 of [BM3], that this is an intertwining operator. The image of $\mathcal{A}(M, \sigma, \nu)$ is the Langlands quotient $L(M, \sigma, \nu)$.

2.9. Let (M, σ, ν) be a Langlands parameter as in section 2.5, and $\nu = Re\ \nu + \sqrt{-1} Im\ \nu$ with $Im\ \nu \neq 0$. Set

$$R_{M_1} = \{\alpha \in R : \langle Im\ \nu, \alpha \rangle = 0, \quad R_{N_1} = \{\alpha \in R : \langle Im\ \nu, \alpha \rangle > 0\}. \tag{2.9.1}$$

Clearly, $\Pi_M \subset R_{M_1}$. Moreover, R_{M_1} is a root subsystem of R. Set $R_{M_1}^+ = R_{M_1} \cap R^+$, and let Δ_{M_1} denote the set of corresponding simple roots. Note that $\Pi_M \subset \Delta_{M_1}$, but Δ_{M_1} need not be a subset of Π. But $\mathbb{H}_M$ is naturally a subalgebra of $\mathbb{H}_{M_1}$.

Assume $L(M, \sigma, \nu)$ is Hermitian and let $w \in W$ be as in proposition 2.8. From $w\nu = -\overline{\nu}$, it follows that $w \in W(M_1)$.

The triple $(M, \sigma, Re\ \nu)$ is a Langlands triple for $\mathbb{H}_{M_1}$, so it makes sense to consider $\sigma_1 = L_{M_1}(M, \sigma, Re\ \nu)$. Moreover, σ_1 is Hermitian in $\mathbb{H}_{M_1}$.

Proposition. *With the notation as above,*

$$I(M_1, (\sigma_1 \otimes \mathbb{C}_{Im\ \nu})) \cong L(M, \sigma, \nu).$$

Proof. It is known that there exists $w' \in W$ (of shortest length) which maps Δ_{M_1} onto a subset $\Pi_{M'}$ of Π. This defines an isomorphism $a_{w'} : \mathbb{H}_{M_1} \to \mathbb{H}_{M'}$, and, similarly to section 2.8, an intertwining operator $\mathcal{A}(M_1 : M', \mathcal{V}, \nu')$. Setting $\mathcal{V} = X_{M_1}(M, \sigma, Re \ \nu)$ and $\nu' = Im \ \nu$, we find that this operator is invertible.

$$X(M, \sigma, \nu) \cong I(M_1, X_{M_1}(M, \sigma, Re \ \nu) \otimes \mathbb{C}_{Im \ \nu}). \qquad (2.9.2)$$

Now, $X(M, \sigma, \nu)$ maps onto $L(M, \sigma, \nu)$ via the operator $\mathcal{A}(M, \sigma, \nu)$ in (2.8.2). On the other hand, $X_{M_1}(M, \sigma, Re \ \nu)$ maps onto σ_1 by the operator $\mathcal{A}_{M_1}(M, \sigma, Re \ \nu)$, and by induction (which is an exact functor) we find a map from the right hand side of (2.9.2) onto $I(M_1, (\sigma_1 \otimes \mathbb{C}_{Im \ \nu})$ whose kernel is $\mathbb{H} \otimes_{\mathbb{H}_{M_1}} (\ker \mathcal{A}_{M_1}(M, \sigma, Re \ \nu)) \cong \ker \mathcal{A}(M', \sigma, \nu)$, which is identical to $\ker \mathcal{A}(M, \sigma, \nu)$.

$\square$

Corollary. *Assuming the previous notation, $L(M, \sigma, \nu)$ is unitary if and only if $L_{M_1}(M, \sigma, Re \ \nu)$ is unitary.*

Proof. This follows immediately from proposition 2.9, and the fact that if χ is a purely imaginary character and $I(M, U \otimes \mathbb{C}_\chi)$ is irreducible, then $I(M, U \otimes \mathbb{C}_\chi)$ is unitary for $\mathbb{H}$ if and only if U is unitary for $\mathbb{H}_M$. $\square$

3. Geometric Hecke algebras

We will restrict to the case of *geometric Hecke algebras*, as in [Lu2]-[Lu4]. We review some facts about these algebras and their geometric classification.

3.1. For an algebraic group $\mathbf{G}$ and a $\mathbf{G}$-variety X, let $H^\bullet_{\mathbf{G}}(X) = H^\bullet_{\mathbf{G}}(X, \mathbb{C})$, respectively $H^{\mathbf{G}}_\bullet(X) = H^{\mathbf{G}}_\bullet(X, \mathbb{C})$ denote the equivariant cohomology, respectively homology (as in section 1 of [Lu2]). The component group of $\mathbf{G}$, $A(\mathbf{G}) = \mathbf{G}/\mathbf{G}^0$ acts naturally on $H^\bullet_{\mathbf{G}^0}(X)$ and $H^{\mathbf{G}^0}_\bullet(X)$. The cup product defines a structure of graded $H^\bullet_{\mathbf{G}}(X)$-module on $H^{\mathbf{G}}_\bullet(X)$. If pt is a point of X, one uses the notation $H^\bullet_{\mathbf{G}} = H^\bullet_{\mathbf{G}}(\{pt\})$, respectively $H^{\mathbf{G}}_\bullet = H^{\mathbf{G}}_\bullet(\{pt\})$. There is a $\mathbb{C}$-algebra homomorphism $H^\bullet_{\mathbf{G}} \to H^\bullet_{\mathbf{G}}(X)$ induced by the map $X \to \{pt\}$, and therefore $H^\bullet_{\mathbf{G}}$ and $H^{\mathbf{G}}_\bullet$ can both be considered as $H^\bullet_{\mathbf{G}}$-modules.

For a subset $\mathcal{S}$ of $\mathbf{G}$, or $\mathfrak{g}$, let $Z_{\mathbf{G}}(\mathcal{S})$, $N_{\mathbf{G}}(\mathcal{S})$ denote the centralizer, respectively the normalizer of $\mathcal{S}$ in $\mathbf{G}$.

Let G be a reductive connected complex algebraic group, with Lie algebra $\mathfrak{g}$. Let $P = LN$ denote a parabolic subgroup, with $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ the corresponding Lie algebras, such that $\mathfrak{l}$ admits an irreducible L-equivariant cuspidal local system (as in [Lu2],[Lu5]) $\mathcal{L}$ on a nilpotent L-orbit $\mathcal{C} \subset \mathfrak{l}$. The classification of cuspidal local systems can be found in [Lu5]. In particular, $W = N(L)/L$ is a Coxeter group.

Let H be the center of L with Lie algebra $\mathfrak{h}$, and let R be the set of nonzero weights α for the *ad*-action of $\mathfrak{h}$ on $\mathfrak{g}$, and $R^+ \subset R$ the set of weights for which the corresponding weight space $\mathfrak{g}_\alpha \subset \mathfrak{n}$. For each parabolic $P_j = L_j N_j$, $j = 1, n$, such that $P \subset P_j$ maximally and $L \subset L_j$, let $R^+_j = \{\alpha \in R^+ : \alpha(\mathfrak{z}(\mathfrak{l}_j)) = 0\}$, where $\mathfrak{z}(\mathfrak{l}_j)$ denotes the center of $\mathfrak{l}_j$. It is shown in [Lu2] that each R^+_j contains a unique α_j such that $\alpha_j \notin 2R$.

Let $Z_G(\mathcal{C})$ denote the centralizer in G of a Lie triple for $\mathcal{C}$, and $\mathfrak{z}(\mathcal{C})$ its Lie algebra.

Proposition ([Lu2])**.**

(a) R is a (possibly non-reduced) root system in $\mathfrak{h}^*$, with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Moreover, W is the corresponding Weyl group.

(b) H is a maximal torus in $Z^0 = Z_G^0(\mathcal{C})$.

(c) W is isomorphic to $W(Z_G^0(\mathcal{C})) = N_{Z^0}(H)/H$.

(d) The set of roots in $\mathfrak{z}(\mathcal{C})$ with respect to $\mathfrak{h}$ is exactly the set of reduced roots in R.

For each $j = 1, n$, let $c_j \geq 2$ be such that

$$ad(e)^{c_j-2} : \mathfrak{l}_j \cap \mathfrak{n} \to \mathfrak{l}_j \cap \mathfrak{n} \neq 0, \text{ and } ad(e)^{c_j-1} : \mathfrak{l}_j \cap \mathfrak{n} \to \mathfrak{l}_j \cap \mathfrak{n} = 0. \qquad (3.1.1)$$

By proposition 2.12 in [Lu2], $c_i = c_j$ whenever α_i and α_j are W-conjugate. Therefore, one can define a Hecke algebra $\mathbb{H}$ as in (2.2.1),(2.2.2). In view of the proposition above, one can think of this algebra as essentially a graded Hecke algebra with unequal parameters for the centralizer $\mathfrak{z}(\mathcal{C})$. The explicit algebras which may appear are listed in 2.13 of [Lu2]. The more familiar case of Hecke algebras with equal parameters arise when one takes P to be a Borel subgroup, and $\mathcal{C}$ and $\mathcal{L}$ to be trivial.

The geometric realization of $\mathbb{H}$ is obtained as follows. Consider the varieties

$$\dot{\mathfrak{g}}_N = \{(x, gP) \in \mathfrak{g} \times G/P \mid Ad(g^{-1})x \in \mathfrak{n}\}, \qquad (3.1.2)$$

$$\ddot{\mathfrak{g}}_N = \{(x, gP, g'P) \in \mathfrak{g} \times G/P \times G/P \mid (x, gP), (x, g'P) \in \dot{\mathfrak{g}}_N\}.$$

Clearly $\ddot{\mathfrak{g}}_N \subset \dot{\mathfrak{g}}_N \times \dot{\mathfrak{g}}_N$. The local system $\mathcal{L}$ gives a local systems $\dot{\mathcal{L}}$ on $\dot{\mathfrak{g}}_N$ and $\ddot{\mathcal{L}}$ on $\ddot{\mathfrak{g}}_N$. When P is a Borel subgroup and $\mathcal{L}$ is trivial, one recovers the classical objects: $\dot{\mathfrak{g}}_N$ is the cotangent bundle of the flag variety of G, and $\ddot{\mathfrak{g}}_N$ is the Steinberg variety.

The group $G \times \mathbb{C}^*$ acts on $\mathfrak{g}$ by $(g_1, \lambda) \cdot x = \lambda^{-2} Ad(g_1)x$, for every $x \in \mathfrak{g}, g_1 \in G, \lambda \in \mathbb{C}^*$. In [Lu2], the vector space $H_\bullet^{G \times \mathbb{C}^*}(\ddot{\mathfrak{g}}_N, \ddot{\mathcal{L}})$ is endowed with left and right actions of W and $S(\mathfrak{h}^* \oplus \mathbb{C}) \cong H_{G \times \mathbb{C}^*}^\bullet(\dot{\mathfrak{g}}_N)$, and it is proved (theorem 6.3 and corollary 6.4) that

$$H_\bullet^{G \times \mathbb{C}^*}(\ddot{\mathfrak{g}}_N, \mathbb{C}) \cong \mathbb{H}, \text{ as } \mathbb{H}\text{-bimodules.} \qquad (3.1.3)$$

3.2.   Now we recall the construction of standard modules for $\mathbb{H}$. Fix a nilpotent element e in $\mathfrak{g}$, and let $\mathcal{P}_e$ be the variety

$$\mathcal{P}_e = \{gP \in G/P : Ad(g^{-1})e \in \mathcal{C} + \mathfrak{n}\}. \qquad (3.2.1)$$

The centralizer $Z_{G \times \mathbb{C}^*}(e)$ acts on $\mathcal{P}_e$ by $(g_1, \lambda).gP = (g_1 g)P$.

[Lu2] constructs actions of W and $S(\mathfrak{h}^* \oplus \mathbb{C})$ on $H_\bullet^{Z_{G \times \mathbb{C}^*}^0(e)}(\mathcal{P}_e, \dot{\mathcal{L}})$, and proves that these are compatible with the relations between the generators of $\mathbb{H}$, therefore obtaining a module of $\mathbb{H}$ (theorem 8.13). The component group $A_{G \times \mathbb{C}^*}(e)$ acts on $H_\bullet^{Z_{G \times \mathbb{C}^*}^0(e)}(\mathcal{P}_e, \dot{\mathcal{L}})$, and commutes with the $\mathbb{H}$-action (8.5).

Consider the variety $\mathcal{V}$ of semisimple $Z_{G \times \mathbb{C}^*}^0(e)$-orbits on the Lie algebra $\mathfrak{z}_{G \times \mathbb{C}^*}(e) = \{(x, r_0) \in \mathfrak{g} \oplus \mathbb{C} : [x, e] = 2r_0 e\}$ of $Z_{G \times \mathbb{C}^*}(e)$. The affine variety $\mathcal{V}$ has $H_{Z_{G \times \mathbb{C}^*}^0(e)}^\bullet$ as the coordinate ring. Define the $\mathbb{H}$-modules

$$X(s, r_0, e) = \mathbb{C}_{(s, r_0)} \otimes_{H_{Z_{G \times \mathbb{C}^*}^0(e)}^\bullet} H_\bullet^{Z_{G \times \mathbb{C}^*}^0(e)}(\mathcal{P}_e, \dot{\mathcal{L}}), \qquad (3.2.2)$$

where $\mathbb{C}_{(s, r_0)}$ denotes the $H_{Z_{G \times \mathbb{C}^*}^0(e)}^\bullet$-module given by the evaluation at $(s, r_0) \in \mathcal{V}$, $H_{Z_{G \times \mathbb{C}^*}^0(e)}^\bullet \to \mathbb{C}$.

Let $A_{G \times \mathbb{C}^*}(e, s, r_0)$ denote the stabilizer of (s, r_0) in $A_{G \times \mathbb{C}^*}(e)$. For each $\psi \in \widehat{A_{G \times \mathbb{C}^*}}(e, s, r_0)$, define

$$X(s, r_0, e, \psi) = \operatorname{Hom}_{A_{G \times \mathbb{C}^*}(e, s, r_0)}[\psi : X(s, r_0, e)]. \qquad (3.2.3)$$

In particular, when $(s, r_0) = \mathbf{0}$, we have (7.2, 8.9 in [Lu2] and 10.12(d) in [Lu3])

$$X(\mathbf{0}, e) \cong H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}}) \text{ as } W \times A_G(e)\text{-modules.} \qquad (3.2.4)$$

Let $\widehat{A_G(e)}^0$ and $\widehat{A_G(e, s)}^0$ denote the set of representations ψ which appear in the $A_G(e)$-module $H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}})$, respectively in the restriction of this module to $A_G(e, s)$.

Theorem ([Lu2],[Lu3]). *Assume $r_0 \neq 0$.*
(a)([Lu2],8.10) $X(s, r_0, e, \psi) \neq 0$ if and only if $\psi \in \widehat{A_G(e, s, r_0)}^0$.
(b)([Lu2],8.15) Any simple $\mathbb{H}$-module on which r acts by r_0 is a quotient $\overline{X}(s, r_0, e, \psi)$ of an $X(s, r_0, e, \psi)$, where $\psi \in \widehat{A_G(e, s)}^0$.
(c)([Lu3],8.18) The set of isomorphism classes of simple $\mathbb{H}$-modules with central character (s, r_0) is in 1-1 correspondence with the set

$$\mathcal{M}_{s, r_0} = Z_G(s)\text{-conjugacy classes on } \{(e, \psi) : [s, e] = 2r_0 e, \ \psi \in \widehat{A_G(e, s)}^0\}. \qquad (3.2.5)$$

3.3. Fix a semisimple class $(s, r_0) \in \mathcal{V}$. The following theorem is immediately implied by the results in [Lu3], 10.5-10.7.

Theorem ([Lu3]). *If a composition factor Y of the standard module $X(s, r_0, e, \psi)$ is parameterized by the class $(e', \psi') \in \mathcal{M}_{s, r_0}$, then necessarily $\mathcal{O} \subset \overline{\mathcal{O}'}$, where $\mathcal{O}$, $\mathcal{O}'$ are the $Z_G(s)$-orbits of e, respectively e' in $\mathcal{M}_{s, r_0}$. Moreover, $\mathcal{O} = \mathcal{O}'$ if and only if $Y = \overline{X}(s, r_0, e, \psi)$.*

This is the geometric equivalent of the classical results for real and p-adic groups concerning the minimality of the Langlands parameter in a standard module (see [BW], IV.4.13 and XI.2.13).

Finally, an immediate corollary of theorem 3.3 (see 8.17 in [Lu2], equivalently 10.9 in [Lu3]) is that, if $\mathcal{O}$ is the unique open $Z_G(s)$-orbit in $\mathcal{M}_{s, r_0}$, then $X(s, r_0, e, \psi)$, $\psi \in \widehat{A_G(e, s)}^0$ is simple.

3.4. An important role in the determination of the unitary dual will be played by the W-structure of standard and simple modules.

The continuation argument in 10.13 in [Lu3] in conjunction with (3.2.4) shows that

$$X(s, r_0, e)|_W \cong H_{\bullet}^{\{1\}}(\mathcal{P}_e, \dot{\mathcal{L}}), \text{ as } W \times A_G(s, e)\text{-representations.} \qquad (3.4.1)$$

(By definition, $H_j^{\{1\}}(X, \mathcal{L}) = H_c^{2 \dim(X) - j}(X, \mathcal{L}^*)^*$, where $H_c^{\bullet}$ denotes the cohomology with compact support, while $*$ denotes the dual vector space or local system.)

Denote

$$\mathcal{N}_G = G\text{-conjugacy classes of pairs } \{(e, \phi) : e \in \mathfrak{g} \text{ nilpotent}, \phi \in \widehat{A_G(e)}\}. \qquad (3.4.2)$$

The *generalized Springer correspondence* ([Lu5]) gives, in particular, an injection

$$\Phi_{\mathcal{C}, \mathcal{L}} : \widehat{W} \hookrightarrow \mathcal{N}_G. \qquad (3.4.3)$$

(Recall that $W = N_G(L)/L$.) The case of the classical Springer correspondence appears when P a Borel subgroup, $\mathcal{C}$ and $\mathcal{L}$ are trivial, and W is the Weyl group of G.

More precisely, let (e, ϕ) be a G-conjugacy class in $\mathcal{N}_G$. Set $\mathcal{O}_e = G \cdot e$, and let ε_ϕ be the local system on $\mathcal{O}_e$ corresponding to ϕ. Then [Lu5] (6.2,6.3) attaches to (e, ϕ) a unique G-conjugacy class $(L', \mathcal{C}', \mathcal{L}')$, where $L' \subset P'$ is a Levi subgroup, $\mathcal{C}'$ a nilpotent L'-orbit in $\mathfrak{l}'$, and $\mathcal{L}'$ is a local system on $\mathcal{C}'$, such that:

(a) $H_c^{dim(\mathcal{O}_e) - dim(\mathcal{C}')}(\mathcal{O}_e \cap (\mathcal{C}' + \mathfrak{n}'), \varepsilon_\phi) \neq 0;$ (3.4.4)

(b) P' is minimal with respect to (a).

The local system $\mathcal{L}'$ on $\mathcal{C}'$ is constructed from ε_ϕ (see [Lu5], 6.2, for the precise definition). (Note that in fact, (3.4.4) gives a definition of cuspidal: one could define $\mathcal{L}'$ to be a cuspidal local system for G if in (3.4.4), $P' = G$.) It is shown in [Lu5] that all $\mathcal{L}'$ appearing in this way must be cuspidal for the corresponding L'.

If we denote by $\mathcal{M}_{\mathcal{C}, \mathcal{L}}$ the subset of $\mathcal{N}_G$ attached to $(L, \mathcal{C}, \mathcal{L})$ by (3.4.4), the generalized Springer correspondence (3.4.3) can be reformulated as a bijection $\mathcal{M}_{\mathcal{C}, \mathcal{L}} \leftrightarrow \widehat{W}$. For $(e, \phi) \in \mathcal{M}_{\mathcal{C}, \mathcal{L}}$, there corresponds an irreducible W-representation, which we will denote by $\mu(\mathcal{O}_e, \phi)$, constructed in $\mathrm{Hom}_{A(e)}[\phi : H_c^\bullet(\mathcal{P}_e, \dot{\mathcal{L}})]$.

With respect to the closure ordering of nilpotent orbits, the smallest orbit, $\mathcal{O}_{min}$, appearing in $\mathrm{Im}\Phi_{\mathcal{C}, \mathcal{L}}$ is $G \cdot \mathcal{C}$, and the largest orbit, $\mathcal{O}_{max}$, is the Lusztig-Spaltenstein induced orbit $\mathrm{Ind}_L^G(\mathcal{C})$. Moreover, each one of these two orbits supports exactly one local system (denoted ϕ_{max} and ϕ_{min}, respectively) which enters the parameterization of $\mathbb{H}$.

The correspondence is normalized such that

$$\mu(\mathcal{O}_{min}, \phi_{min}) = triv \text{ and } \mu(\mathcal{O}_{max}, \phi_{max}) = sgn.$$

3.5.   The previous discussion gives a classification of simple $\mathbb{H}$-modules via *lowest W-types*. This is the analogue of Vogan's classification by lowest K-types for real reductive groups ([Vo2]), and it appeared in the setting of equal parameter affine Hecke algebras in [BM1].

Fix $\psi \in \widehat{A_G(s, e)}$. If $\phi \in \widehat{A_G(e)}^0$ and

$$\mathrm{Hom}_W[(\mu(\mathcal{O}_e, \phi) : X(s, r_0, e, \psi)] \neq 0,$$ (3.5.1)

then we will call $\mu(\mathcal{O}, \phi)$ a *lowest W-type* for $X(s, r_0, e, \psi)$.

Proposition. *The simple module* $\overline{X}(s, r_0, e, \psi)$ *is the unique composition factor of* $X(s, r_0, e, \psi)$ *which contains the lowest W-types* $\mu(\mathcal{O}_e, \phi)$ *with multiplicity* $[\phi \mid_{A_G(s,e)}$ *:* $\psi]$. *Moreover, if* μ' *is a W-type appearing in* $\overline{X}(s, r_0, e, \psi)$, *different than the lowest W-types, then* $\Phi_{\mathcal{C}, \mathcal{L}}(\mu') = (\mathcal{O}', \phi')$, *for some* $\mathcal{O}'$ *such that* $\mathcal{O} \subset \overline{\mathcal{O}'} \setminus \mathcal{O}'$.

Proof. It follows directly from theorem 3.3 and the discussion in section 3.4, particularly, equation (3.4.1). $\square$

3.6.   If $s \in \mathfrak{g}$ is semisimple, let $s = s_{hyp} + s_{ell}$ denote its decomposition into hyperbolic and elliptic parts.

Theorem ([Lu4])**.** *A simple* $\mathbb{H}$-*module* $\overline{X}(s, r_0, e, \psi)$ *is tempered if and only if* $\{s_{hyp}, e\}$ *can be embedded into a Lie triple of* $\mathcal{O}_e$. *In this case* $\overline{X}(s, r_0, e, \psi) = X(s, r_0, e, \psi)$.

If in addition, $\mathcal{O}_e$ is a distinguished nilpotent orbit, then $X(s, r_0, e, \psi)$ is a discrete series.

Note that, if (s, e, ϕ) is the geometric parameter of a real $(s_{ell} = 0)$ tempered simple $\mathbb{H}$-module, $A_G(s, e) = A_G(e)$, and $X(s, r_0, e, \phi)$ has a unique lowest W-type, namely $\mu(\mathcal{O}_e, \phi)$. In addition, $\mu(\mathcal{O}_e, \phi)$ has multiplicity one. We need one more fact, in the notation before (3.4.4):

$$H_c^{dim(\mathcal{O}_e)-dim(\mathcal{C}')}(\mathcal{O}_e \cap (\mathcal{C}' + \mathfrak{n}'), \varepsilon_\phi) \neq 0 \text{ if and only if } H_c^*(\mathcal{O}_e \cap (\mathcal{C}' + \mathfrak{n}'), \varepsilon_\phi) \neq 0.$$
$$(3.6.1)$$

For Borel-Moore homology and the classical Springer correspondence, this fact was proved first in [Sh1] (theorem 2.5) for classical groups and [Sh2] for F_4, [BS] (section 2) for E_6, E_7, E_8, and for G_2 it follows from [Spr2] (7.16). In the generality needed here, it is proved in [Lu7], theorem 24.4.(d). The proof is a consequence of the algorithm for computing Green functions.

Therefore, we have a bijection between $\widehat{W}$ and simple tempered $\mathbb{H}$-modules with real central character, when $\mathbb{H}$ is of geometric type. Combining this with proposition 3.5, we obtain the following result.

Corollary. *Assume $\mathbb{H}$ is of geometric type. The set of simple tempered $\mathbb{H}$-modules with real central character is linearly independent in the Grothendieck group of W.*

Remark. In the Grothendieck group of W, one can consider two sets: one set given by the W-characters of tempered $\mathbb{H}$-modules with real central character, and the other set, the natural basis of irreducible W-representations. The statements above say that the first set is also a basis, and that, in some ordering (coming from the lowest W-type map), the change of bases matrix is uni-triangular. It is natural to expect that this should hold for arbitrary parameters Hecke algebras $\mathbb{H}$. For $\mathbb{H}$ of type G_2 with arbitrary parameters, this correspondence is part of [KR]. When $\mathbb{H}$ is of type B/C, the cases not covered here are all accounted for by the recent paper [Ka] and the geometry of the "exotic nilpotent cone". Since a geometric framework exists, this result will most likely follow, but the "exotic" lowest W-type correspondence may not be compatible with the one constructed for Lusztig's parameters. For multi-laced Hecke algebras with unequal parameters (including F_4), the results of [OS] may also imply this statement.

4. HERMITIAN FORMS

We retain the notation from the previous sections. The graded Hecke algebra $\mathbb{H}$ will be assumed geometric and all parameters and central characters are assumed to be real. We will fix $r_0 = 1/2$ and drop it from the notation. (It is sufficient to determine the unitary dual of $\mathbb{H}$ for one particular value $r_0 \neq 0$.)

4.1. Let (M, σ, ν), ν real, be a Hermitian Langlands parameter as in section 2.8. We constructed the intertwining operator

$$\mathcal{A}(M, \sigma, \nu) : X(M, \sigma, \nu) \to I(M, \sigma \otimes \mathbb{C}_{-\nu}), \ x \otimes (v \otimes 1_\nu) \mapsto xr_w \otimes (a_w(v) \otimes 1_{-\nu}),$$
$$(4.1.1)$$

where w is such that $wM = M$, $a_w : w\sigma \xrightarrow{\cong} \sigma$, and $w\nu = -\nu$.

By the results in the previous section, $X(M, \sigma, \nu)$ contains a special set of W-representations, the lowest W-types. It is an empirical fact that every $L(M, \sigma, \nu)$ contains one lowest W-type μ_0 with multiplicity 1. (This is verified case by case for

the exceptional Hecke algebra, for those of classical types this is automatic from the fact that the component groups $A_G(e)$ of nilpotent elements e are always abelian.)

As a $\mathbb{C}[W]$-module,

$$I(M, \sigma \otimes \mathbb{C}_\nu) \mid_W = \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} (\sigma \mid_W). \tag{4.1.2}$$

For any W-type (μ, V_μ), $\mathcal{A}(M, \sigma, \nu)$ induces an operator

$$r_\mu(M, \sigma, \nu) : \operatorname{Hom}_W(\mu, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} \sigma) \to \operatorname{Hom}_W(\mu, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} \sigma). \tag{4.1.3}$$

By Frobenius reciprocity,

$$\operatorname{Hom}_W(\mu, \mathbb{C}[W] \otimes_{\mathbb{C}[W(M)]} \sigma) \cong \operatorname{Hom}_{W(M)}(\mu, \sigma). \tag{4.1.4}$$

In conclusion, $\mathcal{A}(M, \sigma, \nu)$ gives rise to operators

$$r_\mu(M, \sigma, \nu) : \operatorname{Hom}_{\mathbb{C}[W(M)]}(\mu, \sigma) \to \operatorname{Hom}_{\mathbb{C}[W(M)]}(\mu, \sigma). \tag{4.1.5}$$

The operator $r_{\mu_0}(M, \sigma, \nu)$ is a scalar, and we normalize the intertwining operator $\mathcal{A}(M, \sigma, \nu)$ so that this scalar is 1.

Recall the map $\epsilon : \mathbb{H} \to \mathbb{A}$ (section 2.8). We denote by $\epsilon(x)(\nu)$, the evaluation of an element $\epsilon(x) \in \mathbb{A} = S(\mathfrak{h}^*)$ at $\nu \in \mathfrak{h}$.

Theorem ([BM3]). *Let (M, σ, ν) be a Hermitian Langlands parameter.*

(1) *The map $\mathcal{A}(M, \sigma, \nu)$ is an intertwining operator.*
(2) *The image of the operator $\mathcal{A}(M, \sigma, \nu)$ is $L(M, \sigma, \nu)$ and the Hermitian form on $L(M, \sigma, \nu)$ is given by:*

$$\langle t_x \otimes (v_x \otimes 1_\nu), t_y \otimes (v_y \otimes 1_\nu) \rangle = \langle t_x \otimes (v_x \otimes 1_\nu), t_y r_{w_m} \otimes (a_m(v_y) \otimes 1_{-\nu}) \rangle_h,$$
$$= \langle \epsilon(t_y^* t_x r_{w_m})(\nu) a_m(v_x), v_y \rangle_M.$$

The discussion in this section can be summarized in the following corollary.

Corollary. *A Langlands parameter (M, σ, ν), ν real, is unitary if and only if the following two conditions are satisfied:*

(1) *$w_m M = M$, $w_m V \cong V$, $w_m \nu = -\nu$;*
(2) *the normalized operators $r_\mu(M, \sigma, \nu)$ are positive semidefinite for all $\mu \in \widehat{W}$, such that $\operatorname{Hom}_{W(M)}[\mu : V] \neq 0$.*

One of the main tools for showing $\mathbb{H}$-modules are *not* unitary is to compute the signature of certain $r_\mu(M, \sigma, \nu)$.

4.2.   We consider now the particular case of spherical modules.

Definition. *An $\mathbb{H}$-module V is called spherical if $\operatorname{Hom}_W[V, triv] \neq 0$. It is called generic if $\operatorname{Hom}_W[V, sgn] \neq 0$.*

In the case of Hecke algebra with equal parameters, these definitions are motivated by the correspondence with the representations of split p-adic groups with Iwahori fixed vectors ([BM4, Re2]). In that case, generic $\mathbb{H}$-modules correspond to representations of the p-adic group admitting Whittaker models, and spherical $\mathbb{H}$-modules to representations having fixed vectors under a hyperspecial maximal compact open subgroup. We emphasize that for a split adjoint p-adic group, the unipotent representations admitting Whittaker models have necessarily Iwahori fixed vectors ([Re2]).

The Iwahori-Matsumoto involution, IM, defined by

$$IM(t_w) := (-1)^{l(w)} t_w,$$
$$IM(\omega) := -\omega, \quad \omega \in \mathfrak{h}^*, \tag{4.2.1}$$

interchanges spherical and generic modules. From (2.4.1), it follows that IM commutes with the $*$-operation on $\mathbb{H}$. From (2.4.2), it follows immediately then that IM preserves Hermitian and unitary modules.

Let $X(\nu) = \mathbb{H} \otimes_{\mathbb{A}} \mathbb{C}_\nu$, $\nu \in \mathfrak{h}_{\mathbb{R}}$, be a principal series module. Clearly, $X(\nu) \cong \mathbb{C}[W]$, as W-modules, and in particular, $X(\nu)$ has a unique subquotient containing the *triv* representation of W. Denote this by $L(\nu)$. Let $\mathfrak{h}_{\mathbb{R}}^+$ denote the dominant parameters in $\mathfrak{h}_{\mathbb{R}}$. The Langlands classification and the considerations about intertwining operators presented in the previous sections are summarized as follows.

Proposition. *Assume $\nu \in \mathfrak{h}_{\mathbb{R}}^+$.*
(1) The principal series $X(\nu)$ has a unique irreducible quotient, $L(\nu)$, which is spherical. Moreover, every spherical $\mathbb{H}$-module appears in this way.
(2) A spherical module $L(\nu)$ is Hermitian if and only if $w_0 \nu = -\nu$.
(3) If $w_0 \nu = -\nu$, the image of the intertwining operator

$$A(\nu) : X(\nu) \to X(-\nu), \ A(\nu)(x \otimes 1_\nu) = x r_{w_0} \otimes 1_{-\nu},$$

normalized so that it is $+1$ on the spherical vector, is $L(\nu)$.

In particular, $X(\nu)$ is reducible if and only if $\langle \alpha, \nu \rangle = c_\alpha$, for some $\alpha \in R^+$. For every $\mu \in \widehat{W}$, let

$$r_\mu(\nu) : \mu^* \to \mu^* \tag{4.2.2}$$

be the operator defined by (4.1.5). Let $w_0 = s_1 \cdots s_k$ be a reduced decomposition of w_0 ($k = |R^+|$), where $s_j = s_{\alpha_j}$, $\alpha_j \in \Pi$. The operator $r_\mu(\nu)$ has a decomposition

$$r_\mu(\nu) = r_{\mu, s_1}(s_2 \cdots s_k \nu) \cdot r_{\mu, s_2}(s_3 \cdots s_k \nu) \cdots r_{\mu, s_k}(\nu), \tag{4.2.3}$$

where each $r_{\mu, s_j}(\nu')$ is determined by the equation

$$r_{\mu, s_j}(\nu') = \begin{cases} 1, & \text{on the } (+1)\text{-eigenspace of } s_j \text{ on } \mu^* \\ \frac{c_{\alpha_j} - \nu'}{c_{\alpha_j} + \nu'}, & \text{on the } (-1)\text{-eigenspace of } s_j \text{ on } \mu^*. \end{cases} \tag{4.2.4}$$

4.3. As in corollary 4.1, a spherical Hermitian module $L(\nu)$ is unitary if and only if $r_\mu(\nu)$ are positive semidefinite for all $\mu \in \widehat{W}$.

Definition. *Define the 0-complementary series to be the set $\{\nu \in \mathfrak{h}_{\mathbb{R}}^+ : X(\nu) = L(\nu) \text{ unitary}\}$.*

The 0-complementary series when $\mathbb{H}$ is of type B_n/C_n, with arbitrary unequal parameters, were determined in [BC2]. For type G_2, using the machinery presented in the previous section, in particular the signatures of $r_\mu(\nu)$, it is an easy calculation. We record the result, without proof, next. We use the simple roots

$$\alpha_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}) \text{ and } \alpha_2 = (-1, 1, 0).$$

The Hecke algebra $\mathbb{H}(G_2)$ has $c_{\alpha_1} = 1$ and $c_{\alpha_2} = c > 0$.

Proposition. *Let $\nu = (\nu_1, \nu_1 + \nu_2, -2\nu_1 - \nu_2)$, $\nu_1 \geq 0$, $\nu_2 \geq 0$, be a dominant (spherical) parameter for $\mathbb{H}(G_2)$. There are nine cases depending on the parameter c. The 0-complementary series are as follows.*

(1) $0 < c < 1 : \{3\nu_1 + 2\nu_2 < c\} \cup \{3\nu_1 + \nu_2 > c, \nu_2 < c, 2\nu_1 + \nu_2 < 1\}$.
(2) $c = 1 : \{3\nu_1 + 2\nu_2 < 1\} \cup \{3\nu_1 + \nu_2 > 1, 2\nu_1 + \nu_2 < 1\}$.
(3) $1 < c < \frac{3}{2} : \{3\nu_1 + 2\nu_2 < c\} \cup \{3\nu_1 + \nu_2 > c, 2\nu_1 + \nu_2 < 1\}$.
(4) $c = \frac{3}{2} : \{3\nu_1 + 2\nu_2 < \frac{3}{2}\}$.
(5) $\frac{3}{2} < c < 2 : \{3\nu_1 + 2\nu_2 < c, 2\nu_1 + \nu_2 < 1\}$.
(6) $c = 2 : \{2\nu_1 + \nu_2 < 1\}$.
(7) $2 < c < 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, 3\nu_1 + 2\nu_2 < c\}$.
(8) $c = 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, 3\nu_1 + 2\nu_2 < 3\}$.
(9) $c > 3 : \{2\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_2 > 1, \nu_1 < 1, 3\nu_1 + 2\nu_2 < c\}$.

In addition, the only other unitary spherical representations are the endpoints of the 0-complementary series, and the trivial representation, for which $(\nu_1, \nu_2) = (1, c)$.

In this notation, the Hecke algebras of type G_2 which appear geometrically in [Lu6] are the cases $c = 1$ (equal parameters) and $c = 9$. The latter corresponds to a cuspidal local system on the nilpotent $2A_2$ in E_6.

4.4. We present the unitary dual with real central character for the geometric Hecke algebra of type G_2 in more detail. In the notation as above, this is $\mathbb{H}(G_2, (1, 9))$, but it is better for our parameterization to think of it as the isomorphic $\mathbb{H}(G_2, (3, 1))$. The geometry is attached to the cuspidal local system on $2A_2$ in E_6. The nilpotent orbits $\mathcal{O}$ below and their closure relations are as in [Ca], and the notation for $W(G_2)$-representations is as in [Al].

Proposition. *The unitary irreducible modules with real central character for $\mathbb{H}(G_2, (3, 1))$ are*

Table 1: Unitary dual for $\mathbb{H}(G_2, (3, 1))$

$\mathcal{O}$	**Central character**	**LWT**	**Unitary parameters**
$2A_2$	$(\nu_1, \nu_1 + \nu_2, -2\nu_1 - \nu_2)$	1_1	$\{0 \leq \nu_1, 0 \leq \nu_2, 3\nu_1 + 2\nu_2 \leq 1\} \cup$ $\{0 \leq \nu_1, 3\nu_1 + \nu_2 \geq 1, \nu_2 \geq 1, 2\nu_1 + \nu_2 \leq 3\}$
$2A_2 + A_1$	$(-\frac{1}{2} + \nu, \frac{1}{2} + \nu, -2\nu)$	1_3	$\{0 \leq \nu \leq \frac{3}{2}\}$
A_5	$(3, -\frac{3}{2} + \nu, -\frac{3}{2} - \nu)$	2_2	$\{0 \leq \nu \leq \frac{1}{2}\}$
$E_6(a_3)$	$(1, 2, -3)$	2_1	$D.S.$
$E_6(a_1)$	$(2, 3, -5)$	1_4	$D.S.$
E_6	$(3, 4, -7)$	1_2	St

Here $D.S.$ stands for discrete series, and St for the Steinberg module (which is a discrete series). The proof is by direct computations, similar, but much easier than those in the next section. We don't include the explicit calculations. We remark that the unitary pieces attached to the nilpotent orbits $2A_2 + A_1$ and A_5 match precisely the spherical unitary dual for the Hecke algebra of type A_1 with parameter 3, respectively 1. The centralizers of these two orbits are both of type A_1, so this is consistent with our general philosophy.

To help visualize the answer, we include a picture of the spherical unitary dual in this case, see figure 4.4.
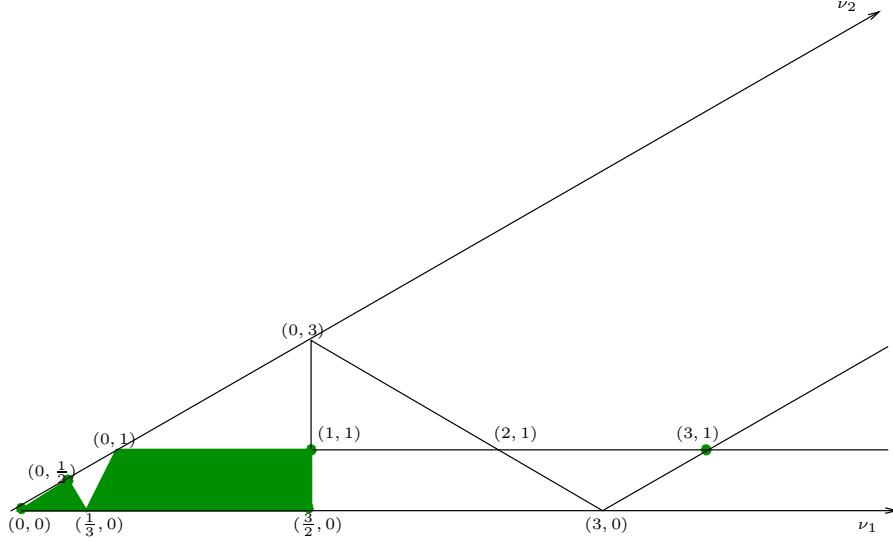
FIGURE 1. Spherical unitary dual for $\mathbb{H}(G_2,(3,1))$

5. AN EXAMPLE: GEOMETRIC HECKE ALGEBRA OF TYPE F_4

We present the classification of the unitary dual for the Hecke algebra $\mathbb{H}$ with unequal parameters of type F_4, which appears geometrically in the classification of [Lu6]. It has labels 1 on the long roots, and 2 on the short roots. We use the following choice for the roots:

$$\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4, \quad \alpha_2 = 2\epsilon_4, \quad \alpha_3 = \epsilon_3 - \epsilon_4, \quad \alpha_4 = \epsilon_2 - \epsilon_3.$$

This algebra is attached to the cuspidal local system on the nilpotent orbit $(3A_1)''$ in E_7. The simple modules are parameterized therefore by a subset of the nilpotent orbits in E_7, larger in the closure ordering than $(3A_1)''$. We list these nilpotent orbits $\mathcal{O}$, each with the corresponding infinitesimal character in $\mathbb{H}$, the W-types coming from the generalized Springer correspondence (LWT), and the centralizer $\mathfrak{z}(\mathcal{O})$ in E_7. The generalized Springer correspondence for this case was computed in [Spa1]. The closure ordering for complex nilpotent orbits in E_7 can be found in [Ca]. (We mention that there is a typographical error in [Ca], the nilpotent orbits $D_6(a_2)$ and $D_5(a_1) + A_1$ are comparable, see figure 2 and [Spa2].)

Table 2: Nilpotent orbits for $(3A_1)''$ in E_7

$\mathcal{O}$	Central character	$\mathfrak{z}(\mathcal{O})$	LWT
$(3A_1)''$	$(\nu_1, \nu_2, \nu_3, \nu_4)$	F_4	1_1
$4A_1$	$(\nu_1, \nu_2, \nu_3, \frac{1}{2})$	C_3	2_3
$A_2 + 3A_1$	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu_1(2,1,1,0) + \nu_2(1,1,0,0)$	G_2	1_3
$(A_3 + A_1)''$	$(0,0,1,-1) + \nu_1(\frac{1}{2}, \frac{1}{2}, 0, 0) + \nu_2(\frac{1}{2}, -\frac{1}{2}, 0, 0)$ $+\nu_3(0, 0, \frac{1}{2}, \frac{1}{2})$	B_3	4_2
$A_3 + 2A_1$	$(\nu_1, 1 + \nu_2, -1 + \nu_2, \frac{1}{2})$	$2A_1$	8_3
$D_4(a_1) + A_1$	$(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$	$2A_1$	$9_1, 2_1$
$A_3 + A_2 + A_1$	$(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{1}{2}) + \nu(2,1,1,0)$	A_1	4_4

Table 2 – continued from previous page

$D_4 + A_1$	$(\nu_1, \nu_2, \frac{5}{2}, \frac{1}{2})$	B_2	9_3
$(A_5)''$	$(\nu_2 + \frac{3\nu_1}{2}, 2 + \frac{\nu_1}{2}, \frac{\nu_1}{2}, -2 + \frac{\nu_1}{2})$	G_2	8_1
$D_5(a_1) + A_1$	$(\frac{3}{2} + \nu, -\frac{3}{2} + \nu, \frac{3}{2}, \frac{1}{2})$	A_1	4_1
$A_5 + A_1$	$(\frac{1}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{9}{4}) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	A_1	6_1
$D_6(a_2)$	$(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	A_1	16_1
$E_7(a_5)$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	1	$12_1, 6_2$
$D_5 + A_1$	$(2 + \nu, -2 + \nu, \frac{5}{2}, \frac{1}{2})$	A_1	2_2
$D_6(a_1)$	$(\nu, \frac{7}{2}, \frac{3}{2}, \frac{1}{2})$	A_1	9_2
$E_7(a_4)$	$(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	1	$4_3, 8_2$
D_6	$(\nu, \frac{9}{2}, \frac{5}{2}, \frac{1}{2})$	A_1	9_4
$E_7(a_3)$	$(\frac{9}{2}, \frac{5}{2}, \frac{1}{2}, \frac{1}{2})$	1	$8_4, 1_2$
$E_7(a_2)$	$(\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	1	4_5
$E_7(a_1)$	$(\frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2})$	1	2_4
E_7	$(\frac{17}{2}, \frac{9}{2}, \frac{5}{2}, \frac{1}{2})$	1	1_4

5.1. Results. The spherical unitary dual is listed in table 4. These are the unitary modules parameterized by $\mathcal{O}_{min} = (3A_1)''$. For the rest of the nilpotents, the unitary sets are presented next. For every nilpotent $\mathcal{O}$ in table 2, with $\{e, h, f\} \subset \mathcal{O}$ a Lie triple, and $\phi \in \widehat{A_G(e)}^0$, let

$$\mathcal{U}(\mathcal{O}, \phi) = \{(\nu, \psi) : s = \frac{h}{2} + \nu, \phi \in \widehat{A_G(s, e)}^0, \mathrm{Hom}_{A(s,e)}[\psi : \phi] \neq 0,$$

$$\overline{X}(s, e, \psi) \text{ is unitary}\} \tag{5.1.1}$$

denote the set of unitary parameters associated to $(\mathcal{O}, \phi)$. The goal is to relate $\mathcal{U}(\mathcal{O}, \phi)$ with the spherical unitary dual $\mathcal{SU}$ of a Hecke algebra of different type.

Theorem. *With the notation as before (particularly table 2 and (5.1.1)), the unitary irreducible modules with real central character for $\mathbb{H}(F_4, (1, 2))$ are:*

Table 3: Unitary dual of $\mathbb{H}(F_4, (1, 2))$

$\mathcal{O}$	$\mu(\mathcal{O}, \phi)$	$\mathcal{U}(\mathcal{O}, \phi)$
E_7	1_4	St
$E_7(a_1)$	2_4	$D.S.$
$E_7(a_2)$	4_5	$D.S.$
$E_7(a_3)$	8_4	$D.S.$
$E_7(a_3)$	1_2	$D.S.$
D_6	9_4	$\mathcal{SU}(A_1, (1))$
$E_7(a_4)$	4_3	$D.S.$
	2_2	$D.S.$
$D_6(a_1)$	9_2	$\mathcal{SU}(A_1, (1))$
$D_5 + A_1$	8_2	$\mathcal{SU}(A_1, (1))$
$E_7(a_5)$	12_1	$D.S.$
	6_2	$D.S.$
$D_6(a_2)$	16_1	$\mathcal{SU}(A_1, (1))$
$A_5 + A_1$	6_1	$\mathcal{SU}(A_1, (1))$
$D_5(a_1) + A_1$	4_1	$0 \leq \nu \leq 1$
$(A_5)''$	8_1	$\mathcal{SU}(G_2, (1, 1))$

Table 3 – continued from previous page

$D_4 + A_1$	9_3	$\mathcal{SU}(B_2, (1, \frac{3}{2}))$
$A_3 A_2 A_1$	4_4	$\mathcal{SU}(A_1, (2))$
$D_4(a_1) + A_1$	9_1	$\{0 \le \nu_2 < \nu_1 \le 3 - \nu_2, \ \nu_1 \le \frac{5}{2}\} \cup \{0 \le \nu_2 = \nu_1 \le \frac{3}{2}\}$
	2_1	$\{0 \le \nu_2 < \nu_1 \le 3 - \nu_2, \ \nu_1 \le \frac{5}{2}\} \cup \{0 \le \nu_2 = \nu_1 \le \frac{5}{2}\}$
$A_3 + 2A_1$	8_3	$\{0 \le \nu_1 \le \frac{1}{2}, \ \nu_1 + 2\nu_2 \le \frac{3}{2}\} \cup$
		$\{0 \le \nu_1 \le \frac{1}{2}, \ 0 \le \nu_2 \le \frac{3}{2}, \ 2\nu_2 - \nu_1 \ge \frac{3}{2}\}$
$(A_3 + A_1)''$	4_2	$\mathcal{SU}(B_3, (1,1)) \cup \{(1 + \nu, \nu, -1 + \nu) : 1 < \nu < 2\}$
$A_2 + 3A_1$	1_3	$\mathcal{SU}(G_2, (2,1)) \cup \{\nu_1 + \nu_2 = 1, \ 0 < \nu_2 < 1\} \cup$
		$\{\nu_1 = 1, \ 0 \le \nu_2 \le 1\} \cup \{3\nu_1 + \nu_2 = 2, \ \frac{1}{2} < \nu_2 \le 2\}$
$4A_1$	2_3	$\{\nu_1 + \nu_2 + \nu_3 \le \frac{3}{2}\} \cup \{\nu_1 + \nu_2 - \nu_3 \ge \frac{3}{2}, \nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}$
		$\cup \{(\nu_1, \nu_2, \frac{3}{2}) : \nu_1 + \nu_2 \le 3, \nu_1 \le \frac{5}{2}\}$
		$\cup \{(\nu_1, 1 + \nu_2, -1 + \nu_2) : 0 \le \nu_1 \le \frac{1}{2}, 0 \le \nu_2 \le \frac{1}{2}\}$
		$\cup \{(\nu_1, 1 + \nu_2, -1 + \nu_2) : \nu_1 + 2\nu_2 = \frac{3}{2}, \frac{1}{2} \le \nu_1 \le \frac{3}{2}\}$
		$\cup \{(\nu, \frac{3}{2}, \frac{1}{2}) : 0 \le \nu \le \frac{5}{2}\} \cup \{(1 + \nu, -1 + \nu, \frac{3}{2}) : 0 \le \nu < \frac{3}{2}\}$
		$\cup \{(2 + \nu, \nu, -2 + \nu) : 0 \le \nu < \frac{1}{2}\} \cup \{(\nu, \frac{7}{2}, \frac{3}{2}) : 0 \le \nu < \frac{1}{2}\}$
		$\cup \{(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})\} \cup \{(\frac{7}{2}, \frac{3}{2}, \frac{1}{2})\} \cup \{(\frac{13}{2}, \frac{7}{2}, \frac{3}{2})\}$
$(3A_1)''$	1_1	$\mathcal{SU}(F_4, (1,2))$, *see table 4*

We will present, in detail, in the next two sections, the calculations giving the unitary dual of $\mathbb{H}(F_4, (1, 2))$. The discussion is organized by nilpotent orbits.

For the distinguished orbits, there is nothing to do. They parameterize discrete series, and the lowest W-types and infinitesimal characters can be read from tables 2 and 3.

5.2. **Maximal parabolic cases.** First we treat the cases of those nilpotent orbits which, in the Bala-Carter classification, correspond to Levi subalgebras of maximal parabolic subalgebras. In terms of the (classical) Langlands classification, they correspond to the induced modules from discrete series on Levi components of maximal parabolic subalgebras. For any Hecke algebra with arbitrary parameters, the Steinberg module St is a discrete series. In this case, our calculation is more general, and we record it for general parameter c. The classification of discrete series depends in general of the particular value of c; for the non-Steinberg discrete series, we restrict to the case $c = 2$.

Proposition. *The Langlands quotient $\overline{X}(M, St, \nu)$ is unitary if and only if:*

Levi M	Nilpotent $\mathcal{O}$ [1]	Central character χ	Unitary
C_3	D_6	$(\nu, 2c + \frac{1}{2}, c + \frac{1}{2}, \frac{1}{2})$	$0 \le \nu \le |c - \frac{3}{2}|$
B_3	$D_5 + A_1$	$(\frac{c}{2} + 1 + \nu, -\frac{c}{2} - 1 + \nu, c + \frac{1}{2}, \frac{1}{2})$	$0 \le \nu \le |\frac{c-3}{2}|$
$\widetilde{A_2} + A_1$	$A_5 + A_1$	$(\frac{1}{4}, c - \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} - c) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$0 \le \nu \le \min\{\frac{1}{2}, |2c - \frac{3}{2}|\}$
$A_2 + \widetilde{A_1}$	$A_3 A_2 A_1$	$(\frac{1}{2}, \frac{c-1}{2}, -\frac{c+1}{2}, \frac{1}{2}) + \nu(2, 1, 1, 0)$	$0 \le \nu \le \min\{\frac{c}{2}, |\frac{3c-2}{2}|\}$

In addition, we have the following complementary series for $\overline{X}(M, \sigma, \nu)$, where $\sigma \ne St$ is a discrete series:

Levi M	Nilpotent $\mathcal{O}$	Central character χ	Unitary
C_3	$D_6(a_1)$	$(\nu, \frac{7}{2}, \frac{3}{2}, \frac{1}{2})$	$0 \le \nu \le \frac{1}{2}$
	$D_6(a_2)$	$(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	$0 \le \nu \le \frac{1}{2}$
B_3	$D_5(a_1) + A_1$	$(\frac{3}{2} + \nu, -\frac{3}{2} + \nu, \frac{3}{2}, \frac{1}{2})$	$0 \le \nu \le 1$

[1] when $c = 2$.

$$E_7$$
$$|$$
$$E_7(a_1)$$
$$|$$
$$E_7(a_2)$$
$$|$$
$$E_7(a_3)$$
$$|$$
$$D_6$$
$$|$$
$$E_7(a_4)$$

$$D_5 + A_1 \qquad\qquad D_6(a_1)$$

$$E_7(a_5)$$
$$|$$
$$D_6(a_2)$$

$$A_5 + A_1 \qquad\qquad D_5(a_1) + A_1$$
$$|$$
$$D_4 + A_1$$
$$(A_5)" \qquad\qquad\qquad |$$
$$A_3 + A_2 + A_1$$

$$D_4(a_1) + A_1$$
$$|$$
$$A_3 + 2A_1$$

$$(A_3 + A_1)" \qquad\qquad A_2 + 3A_1$$

$$4A_1$$
$$|$$
$$(3A_1)"$$
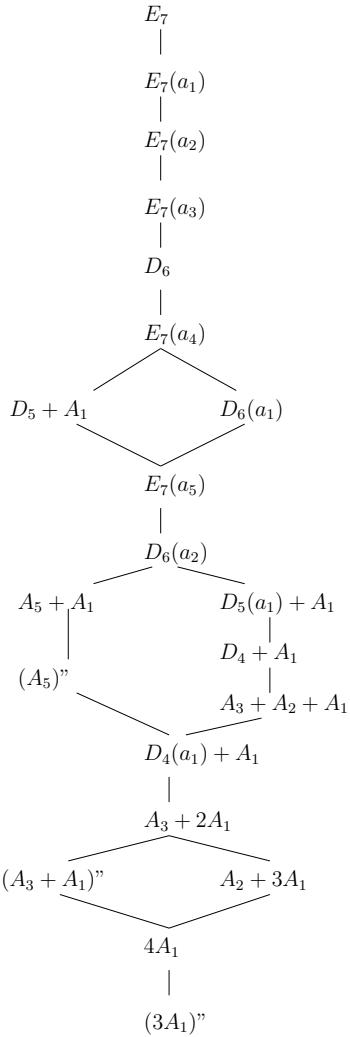
FIGURE 2. Nilpotent orbits parameterizing the $\mathbb{H}(F_4, (1, 2))$-modules.

Proof. The proof is case by case. We compute the intertwining operators for each standard module restricted to every W-type. For the necessary calculations, we constructed matrix realizations (with rational entries) for all irreducible representations of the Weyl group of F_4, and used the software "Mathematica" to multiply matrices in the intertwining operators.

Note that in order to determine the unitarity of the Langlands quotient, we would not need *all* these intertwining operators, but the byproduct of this calculation is that we obtain the W-structure of various irreducible subquotients which appear in the standard modules at reducibility points, in particular (but most importantly) the W-structure of the discrete series. (This is the Jantzen-type filtration from [Vo1].)

$\mathbf{D_6}$: The infinitesimal character is $(\nu, \frac{9}{2}, \frac{5}{2}, \frac{1}{2})$, the lowest W-type is 9_4, and $\mathfrak{z} = A_1$. The standard module is induced from the Steinberg representation on the Hecke subalgebra of type C_3. The W-structure is

$$X(D_6)|_W = 9_4 + 8_4 + 4_5 + 2_4 + 1_4.$$

We calculate the intertwining operators for arbitrary parameter c. In that case, the infinitesimal character is $(\nu, 2c + \frac{1}{2}, c + \frac{1}{2}, \frac{1}{2})$.

$$9_4 : 1; \quad 8_4 : \frac{c - \frac{3}{2} - \nu}{c - \frac{3}{2} + \nu}; \quad 4_5 : \frac{c + \frac{3}{2} - \nu}{c + \frac{3}{2} + \nu}; \quad 2_4 : \frac{(c - \frac{3}{2} - \nu)(3c - \frac{1}{2} - \nu)}{(c - \frac{3}{2} + \nu)(3c - \frac{1}{2} + \nu)};$$

$$1_4 : \frac{(c + \frac{3}{2} - \nu)(3c + \frac{5}{2} - \nu)}{(c + \frac{3}{2} + \nu)(3c + \frac{5}{2} + \nu)}. \tag{5.2.1}$$

In particular, when $c = 2$, the calculation implies that $\overline{X}(D_6, \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$.

$\mathbf{D_6(a_1)}$: The infinitesimal character is $(\nu, \frac{7}{2}, \frac{3}{2}, \frac{1}{2})$, the lowest W-type is 9_2, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from a one-dimensional discrete series in C_3. The W-structure is

$$X(D_6(a_1))|_W = \text{Ind}_{W(C_3)}^{W(F_4)}(111 \times 0) = 9_2 + 4_3 + 8_4 + 1_2 + 2_4.$$

The intertwining operators in this case give:

$$9_2 : 1; \quad 4_3 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}; \quad 8_4 : \frac{\frac{7}{2} - \nu}{\frac{7}{2} + \nu}; \quad 1_2 : \frac{(\frac{1}{2} - \nu)(\frac{7}{2} - \nu)}{(\frac{1}{2} + \nu)(\frac{7}{2} + \nu)};$$

$$2_4 : \frac{(\frac{7}{2} - \nu)(\frac{13}{2} - \nu)}{(\frac{7}{2} + \nu)(\frac{13}{2} + \nu)}. \tag{5.2.2}$$

This indicates that $\overline{X}(D_6(a_1), \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_6(a_1), \frac{1}{2}) = \overline{X}(D_6(a_1), \frac{1}{2}) + X(E_7(a_4), \phi_1);$$

$$\overline{X}(D_6(a_1), \frac{1}{2}) = 9_2 + 8_4 + 2_4.$$

$\mathbf{D_5 + A_1}$: The infinitesimal character is $(2 + \nu, -2 + \nu, \frac{5}{2}, \frac{1}{2})$, the lowest W-type is 8_2, and $\mathfrak{z} = A_1$. The standard module is induced from the Steinberg representation on the Hecke subalgebra of type B_3. The W-structure is

$$X(D_5 + A_1)|_W = 8_2 + 2_2 + 9_4 + 4_5 + 1_4.$$

We calculate the intertwining operators for arbitrary parameter c. In that case, the infinitesimal character is $(\frac{c}{2} + 1 + \nu, -\frac{c}{2} - 1 + \nu, c + \frac{1}{2}, \frac{1}{2})$.

$$8_2 : 1; \quad 2_2 : \frac{\frac{c-3}{2} + \nu}{\frac{c-3}{2} - \nu}; \quad 9_4 : \frac{\frac{3c-1}{2} - \nu}{\frac{3c-1}{2} + \nu}; \quad 4_5 : \frac{(\frac{3c-1}{2} - \nu)(\frac{3c+1}{2} - \nu)}{(\frac{3c-1}{2} + \nu)(\frac{3c+1}{2} + \nu)};$$

$$1_4 : \frac{(\frac{3c-1}{2} - \nu)(\frac{3c+1}{2} - \nu)(\frac{5c+3}{2} - \nu))}{(\frac{3c-1}{2} + \nu)(\frac{3c+1}{2} + \nu)(\frac{5c+3}{2} + \nu)}. \tag{5.2.3}$$

In particular, if $c = 2$, $\overline{X}(D_5 + A_1, \nu)$ is unitary if and only if $0 \le \nu \le \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_5 + A_1, \frac{1}{2}) = \overline{X}(D_5 + A_1, \frac{1}{2}) + X(E_7(a_4), \phi_2)$$

$$\overline{X}(D_5 + A_1, \frac{1}{2})|_W = 8_2 + 9_4 + 4_5 + 1_4.$$

$\mathbf{D_6(a_2)}$: The infinitesimal character is $(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$, the lowest W-type is 16_1, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from a four dimensional discrete series on C_3. The W-structure is

$$X(D_6(a_2))|_W = \mathrm{Ind}_{W(C_3)}^{W(F_4)}(1 \times 11 + 0 \times 111)$$
$$= 16_1 + 2_4 + 9_2 + 2 \cdot 9_4 + 2 \cdot 8_4 + 12_1 + 8_2 + 2 \cdot 4_5 + 6_2 + 1_4.$$

The intertwining operators are (for the W-types with multiplicity greater than 1, we only give the determinant):

$$16_1 : 1; \quad 12_1 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}; \quad 6_2 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}; \quad 2_4 : \frac{(\frac{5}{2} - \nu)(\frac{9}{2} - \nu)}{(\frac{5}{2} + \nu)(\frac{9}{2} + \nu)}; \quad 8_2 : \frac{(\frac{1}{2} - \nu)(\frac{5}{2} - \nu)}{(\frac{1}{2} + \nu)(\frac{5}{2} + \nu)};$$

$$1_4 : \frac{(\frac{1}{2} - \nu)(\frac{5}{2} - \nu)(\frac{9}{2} - \nu)(\frac{11}{2} - \nu)}{(\frac{1}{2} + \nu)(\frac{5}{2} + \nu)(\frac{9}{2} + \nu)(\frac{11}{2} + \nu)}; \quad 9_2 : \frac{\frac{5}{2} - \nu}{\frac{5}{2} + \nu}; \quad 8_4 : Det = \frac{(\frac{1}{2} - \nu)(\frac{5}{2} - \nu)(\frac{9}{2} - \nu)}{(\frac{1}{2} + \nu)(\frac{5}{2} + \nu)(\frac{9}{2} + \nu)};$$

$$4_5 : Det = \frac{(\frac{1}{2} - \nu)^2(\frac{5}{2} - \nu)(\frac{9}{2} - \nu)(\frac{11}{2} - \nu)}{(\frac{1}{2} + \nu)^2(\frac{5}{2} + \nu)(\frac{9}{2} + \nu)(\frac{11}{2} + \nu)}. \quad (5.2.4)$$

It follows that $\overline{X}(D_6(a_2), \nu)$ is unitary if and only if $0 \le \nu \le \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(D_6(a_2), \frac{1}{2}) = \overline{X}(D_6(a_2), \frac{1}{2}) + X(E_7(a_5), \phi_1) + X(E_7(a_5), \phi_2);$$

$$\overline{X}(D_6(a_2), \frac{1}{2})|_W = 16_1 + 2_4 + 9_2 + 8_4 + 9_4.$$

$\mathbf{A_5 + A_1}$: The infinitesimal character is $(\frac{1}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{9}{4}) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the lowest W-type is 6_1, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_2 + A_1$.

$$X(A_5 + A_1)|_W = \mathrm{Ind}_{W(\widetilde{A}_2 + A_1)}^{W(F_4)}((111) \otimes (11))$$
$$= 6_1 + 16_1 + 2 \cdot 8_4 + 8_2 + 4_3 + 4_5 + 12_1 + 9_2 + 2 \cdot 9_4 + 2_4 + 1_4.$$

We compute the intertwining operator for arbitrary parameter c. In that case, the infinitesimal character is $(\frac{1}{4}, c - \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} - c) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

$$6_1 : 1; \quad 16_1 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}; \quad 8_2 : \frac{(\frac{1}{2} - \nu)^2(\frac{3}{2} - \nu)}{(\frac{1}{2} + \nu)^2(\frac{3}{2} + \nu)}; \quad 12_1 : \frac{(\frac{1}{2} - \nu)^2}{(\frac{1}{2} + \nu)^2}; \quad 9_2 : \frac{(\frac{1}{2} - \nu)(2c - \frac{3}{2} - \nu)}{(\frac{1}{2} + \nu)(2c - \frac{3}{2} + \nu)};$$

$$\tag{5.2.5}$$

$$4_3 : \frac{2c - \frac{3}{2} - \nu}{2c - \frac{3}{2} + \nu}; \quad 2_4 : \frac{(\frac{1}{2} - \nu)(c + \frac{1}{2} - \nu)(2c - \frac{3}{2} - \nu)(2c + \frac{1}{2} - \nu)}{(\frac{1}{2} + \nu)(c + \frac{1}{2} + \nu)(2c - \frac{3}{2} + \nu)(2c + \frac{1}{2} + \nu)};$$

$$4_5 : \frac{(\frac{1}{2} - \nu)^2(\frac{3}{2} - \nu)(c + \frac{1}{2} - \nu)(2c + \frac{1}{2} - \nu)}{(\frac{1}{2} + \nu)^2(\frac{3}{2} + \nu)(c + \frac{1}{2} + \nu)(2c + \frac{1}{2} + \nu)};$$

$$1_4 : \frac{(\frac{1}{2} - \nu)^2(\frac{3}{2} - \nu)(c + \frac{1}{2} - \nu)(2c + \frac{1}{2} - \nu)(2c + \frac{3}{2} - \nu)}{(\frac{1}{2} + \nu)^2(\frac{3}{2} - \nu)(c + \frac{1}{2} + \nu)(2c + \frac{1}{2} + \nu)(2c + \frac{3}{2} + \nu)};$$

$$8_4 : Det = \frac{(\frac{1}{2} - \nu)^3(c + \frac{1}{2} - \nu)(2c - \frac{3}{2} - \nu)(2c + \frac{1}{2} - \nu)}{(\frac{1}{2} + \nu)^3(c + \frac{1}{2} + \nu)(2c - \frac{3}{2} + \nu)(2c + \frac{1}{2} + \nu)};$$

$$9_4 : Det = \frac{(\frac{1}{2} - \nu)^3(\frac{3}{2} - \nu)(c + \frac{1}{2} - \nu)(2c + \frac{1}{2} - \nu)}{(\frac{1}{2} + \nu)^3(\frac{3}{2} + \nu)(c + \frac{1}{2} + \nu)(2c + \frac{1}{2} + \nu)}. \tag{5.2.6}$$

It follows that in the case $c = 2$, $\overline{X}(A_5 + A_1, \nu)$ is unitary if and only if $0 \leq \nu \leq \frac{1}{2}$. At $\nu = \frac{1}{2}$, the standard module decomposes as

$$X(A_5 A_1, \frac{1}{2}) = \overline{X}(A_5 A_1, \frac{1}{2}) + \overline{X}(D_6(a_2), \frac{1}{2}) + X(E_7(a_5), \phi_1)$$

$$\overline{X}(A_5 A_1, \frac{1}{2}) = 6_1 + 4_3.$$

$\mathbf{D_5(a_1) + A_1}$: The infinitesimal character is $(\frac{3}{2} + \nu, -\frac{3}{2} + \nu, \frac{3}{2}, \frac{1}{2})$, the lowest W-type is 4_1, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from a two-dimensional discrete series on B_3. The W-structure is

$$X(D_5(a_1) + A_1))|_W = \mathrm{Ind}_{W(B_3)}^{W(F_4)}(0 \times 12) = 4_1 + 16_1 + 2_4 + 9_4 + 9_2 + 8_4.$$

The intertwining operators are:

$$4_1 : 1; \quad 16_1 : \frac{1 - \nu}{1 + \nu}; \quad 9_2 : \frac{(1 - \nu)(2 - \nu)}{(1 + \nu)(2 + \nu)}; \quad 9_4 : \frac{(1 - \nu)(4 - \nu)}{(1 + \nu)(4 + \nu)};$$

$$8_4 : \frac{(1 - \nu)(2 - \nu)(4 - \nu)}{(1 + \nu)(2 + \nu)(4 + \nu)}; \quad 2_4 : \frac{(1 - \nu)(2 - \nu)(4 - \nu)(5 - \nu)}{(1 + \nu)(2 + \nu)(4 + \nu)(5 + \nu)}. \tag{5.2.7}$$

It follows that $\overline{X}(D_5(a_1) + A_1)$ is unitary if and only if $0 \leq \nu \leq 1$. At $\nu = 1$, the standard module decomposes as

$$X(D_5(a_1) + A_1, 1) = \overline{X}(D_5(a_1) + A_1, 1) + \overline{X}(D_6(a_2), \frac{1}{2})$$

$$\overline{X}(D_5(a_1) + A_1, 1)|_W = 4_1.$$

$\mathbf{A_3 + A_2 + A_1}$: The infinitesimal character is $(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{1}{2}) + \nu(2, 1, 1, 0)$, the lowest W-type is 4_4, and $\mathfrak{z} = A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on $A_2 + \widetilde{A}_1$. The W-structure is:

$$X(A_3 + A_2 + A_1)|_W = \mathrm{Ind}_{W(A_2 + \widetilde{A}_1)}^{W(F_4)}((111) \otimes (11))$$

$$= 4_4 + 16_1 + 8_4 + 2 \cdot 8_2 + 4_5 + 12_1 + 6_1 + 9_3 + 2 \cdot 9_4 + 2_2 + 1_4.$$

We compute the intertwining operators for general parameter c. In that case the infinitesimal character is $(\frac{1}{2}, \frac{c-1}{2}, -\frac{c+1}{2}, \frac{1}{2}) + \nu(2, 1, 1, 0)$.

$4_4 : 1;\ \ 9_3 : \dfrac{\frac{c}{2} - \nu}{\frac{c}{2} + \nu};\ \ 6_1 : \dfrac{\frac{3c-2}{2} - \nu}{\frac{3c-2}{2} + \nu};\ \ 16_1 : \dfrac{(\frac{c}{2} - \nu)(\frac{3c-2}{2} - \nu)}{(\frac{c}{2} + \nu)(\frac{3c-2}{2} + \nu)};$

$12_1 : \dfrac{(\frac{c}{2} - \nu)^2(\frac{3c-2}{2} - \nu)}{(\frac{c}{2} + \nu)^2(\frac{3c-2}{2} + \nu)};\ \ 8_4 : \dfrac{(\frac{c}{2} - \nu)^2(\frac{3c-2}{2} - \nu)(\frac{3c}{2} - \nu)}{(\frac{c}{2} + \nu)^2(\frac{3c-2}{2} + \nu)(\frac{3c}{2} + \nu)};\ \ 2_2 : \dfrac{(\frac{c}{2} - \nu)(\frac{c+1}{2} - \nu)(\frac{c+2}{2} - \nu)}{(\frac{c}{2} + \nu)(\frac{c+1}{2} + \nu)(\frac{c+2}{2} + \nu)};$

$4_5 : \dfrac{(\frac{c}{2} - \nu)^2(\frac{c+1}{2} - \nu)(\frac{c+2}{2} - \nu)(\frac{3c-2}{2} - \nu)(\frac{3c}{2} - \nu)}{(\frac{c}{2} + \nu)^2(\frac{c+1}{2} + \nu)(\frac{c+2}{2} + \nu)(\frac{3c-2}{2} + \nu)(\frac{3c}{2} + \nu)};$

$14_4 : \dfrac{(\frac{c}{2} - \nu)^2(\frac{c+1}{2} - \nu)(\frac{c+2}{2} - \nu)(\frac{3c-2}{2} - \nu)(\frac{3c}{2} - \nu)(\frac{3c+2}{2} - \nu)}{(\frac{c}{2} + \nu)^2(\frac{c+1}{2} + \nu)(\frac{c+2}{2} + \nu)(\frac{3c-2}{2} + \nu)(\frac{3c}{2} + \nu)(\frac{3c+2}{2} + \nu)};$

$8_2 : Det = \dfrac{(\frac{c}{2} - \nu)^3(\frac{c+1}{2} - \nu)(\frac{c+2}{2} - \nu)(\frac{3c-2}{2} - \nu)}{(\frac{c}{2} + \nu)^3(\frac{c+1}{2} + \nu)(\frac{c+2}{2} + \nu)(\frac{3c-2}{2} + \nu)};$

$9_4 : Det = \dfrac{(\frac{c}{2} - \nu)^3(\frac{c+1}{2} - \nu)(\frac{c+2}{2} - \nu)(\frac{3c-2}{2} - \nu)^2(\frac{3c}{2} - \nu)}{(\frac{c}{2} + \nu)^3(\frac{c+1}{2} + \nu)(\frac{c+2}{2} + \nu)(\frac{3c-2}{2} + \nu)^2(\frac{3c}{2} + \nu)}.$ (5.2.8)

In the case $c = 2$, it follows that $\overline{X}(A_3 + A_2 + A_1, \nu)$ is unitary if and only if $0 \le \nu \le 1$. At $\nu = 1$, the standard module decomposes as

$$X(A_3 A_2 A_1, 1) = \overline{X}(A_3 A_2 A_1, 1) + \overline{X}(D_4 A_1, (\frac{3}{2}, \frac{1}{2})) + X(E_7(a_5), \phi_1);$$
$$\overline{X}(A_3 A_2 A_1, 1)|_W = 4_4 + 6_1.$$

Note that $\overline{X}(A_3 A_2 A_1, 1)$ is the IM-dual of $\overline{X}(A_5 A_1, \frac{1}{2})$.

$\square$

5.3. Matching of intertwining operators.

Let (M, σ, ν) be a (Hermitian) Langlands parameter, where σ is a discrete series module for $\mathbb{H}_M$ (or rather $\mathbb{H}_{M_0}$, notation as in section 2.5). We construct a Hecke algebra $\mathbb{H}_\sigma$ with possibly unequal parameters, with a Weyl group W_σ. Let $\{e, h, f\}$ be a Lie triple parameterizing the tempered module σ. Then in all cases which we need to consider except for the case $\{e, h, f\} \subset D_4(a_1) + A_1$, the root system for $\mathbb{H}_\sigma$ and W_σ is the root system of the reductive Lie algebra $\mathfrak{z}(e, h, f)$ of the centralizer in $\mathfrak{g}$ of $\{e, h, f\}$. In the case $D_4(a_1) + A_1$, the centralizer is of type $2A_1$, but the root system we consider is of type B_2. The reason is the presence of a component group $\mathbb{Z}/2\mathbb{Z}$ in that case.

Firstly, let us collect the relevant results from the operators calculations in the maximal parabolic case, in the previous section. For the case of $\mathbb{H}(F_4, (1, 2))$, each generalized discrete series induced from maximal parabolics has a unique lowest W-type, μ_0.

Lemma. *For every maximal parabolic case (M, σ), with lowest W-type μ_0, there is a W-type μ_1 such that*

 (i) *the operator $r_{\mu_1}(M, \sigma, \nu)$ is identical to the operator $r_{sgn}(\nu)$ for a Hecke algebra of type A_1, and*

 (ii) *the Langlands quotient $\overline{X}(M, \sigma, \nu)$ is unitary if and only if $r_{\mu_1}(M, \sigma, \nu)$ is positive semidefinite.*

Proof. The explicit operators and cases are

$\square$

Levi M	$\mathcal{O}$	μ_0	μ_1	$r_{\mu_1}(M,\sigma,\nu)$
C_3	D_6	9_4	8_4	$\frac{1/2-\nu}{1/2+\nu}$
	$D_6(a_1)$	9_2	4_3	$\frac{1/2-\nu}{1/2+\nu}$
	$D_6(a_2)$	16_1	12_1 or 6_2	$\frac{1/2-\nu}{1/2+\nu}$
B_3	D_5+A_1	8_2	2_2	$\frac{1/2-\nu}{1/2+\nu}$
	$D_5(a_1)+A_1$	4_1	16_1	$\frac{1-\nu}{1+\nu}$
$\widetilde{A}_2+A_1$	A_5+A_1	6_1	16_1	$\frac{1/2-\nu}{1/2+\nu}$
$A_2+\widetilde{A}_1$	$A_3+A_2+A_1$	4_4	9_3	$\frac{1-\nu}{1+\nu}$

Definition. *Let $\mathbb{H}$ be an arbitrary graded Hecke algebra. For every (M,σ) maximal parabolic case, we call σ-petite types, the following W-types which appear in $X(M,\sigma,\nu)$: the lowest W-types, and any μ_1 which satisfy the properties from lemma 5.3. Now assume (M,σ) is not maximal parabolic. A W-type μ is called σ-petite, if for every Levi M' such that $M \subset M'$ is maximal, the restriction of μ to $W(M')$ contains only σ-petite $W(M')$-types.*

The main idea is summarized in the next statement, which is proposition 5.6 in [BC1] (see also section 2.10 in [Ci2]). Recall from (1.3.2) that to every W-type μ which appears in $X(M,\sigma,\nu)$ we attach a W_σ-type $\rho(\mu)$.

Proposition. *With the notation above, the set of σ-petite W-types $\{\mu_0,\mu_1,\ldots,\mu_l\}$ in $X(M,\sigma,\nu)$ with the corresponding W_σ-types $\{\rho(\mu_0),\rho(\mu_1),\ldots,\rho(\mu_l)\}$, has the property that*

$$r_\mu(M,\sigma,\nu) = r_{\rho(\mu)}(\nu), \text{ for all } \mu \in \{\mu_0,\ldots,\mu_l\},$$

where $r_\mu(M,\sigma,\nu)$ is the operator in $\mathbb{H}$ (defined as in (4.1.5)), and $r_{\rho(\mu)}$ is the spherical operator in $\mathbb{H}_\sigma$.

The explicit calculations and details will occupy the rest of this section. The matching of intertwining operators from proposition 5.3 is used to rule out non-unitary modules. Ideally, one would like that the remaining modules are all unitary. This is not the case in general, and more (ad-hoc) arguments are needed to prove non-unitarity.

To prove unitarity, we use deformations of unitary irreducible modules (*i.e.*, complementary series methods), calculations of composition factors and the Iwahori-Matsumoto involution.

$(\mathbf{A_5})''$: The infinitesimal character is $(\nu_2 + \frac{3\nu_1}{2}, 2 + \frac{\nu_2}{2}, \frac{\nu_1}{2}, -2 + \frac{\nu_1}{2})$, the lowest W-type is 8_1, and $\mathfrak{z} = G_2$. In the Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_2$. The W-structure is

$$X((A_5)'')|_W = \mathrm{Ind}_{W(\widetilde{A}_2)}^{W(F_4)}((111)).$$

The restrictions of the nearby W-types are as follows:

Nilpotent	$(A_5)''$	A_5A_1	$D_6(a_2)$	$E_7(a_5)$	$E_7(a_5)$
W-type	8_1	6_1	16_1	12_1	6_2
Multiplicity	1	1	2	2	1
$\widetilde{A}_2 \subset \widetilde{A}_2+A_1$	(2)	(11)	(2), (11)	(2), (11)	(2)
$\widetilde{A}_2 \subset C_3$	11×1	11×1	$11 \times 1, 1 \times 11$	$11 \times 1, 1 \times 11$	1×11
$W(\mathfrak{z}) = W(G_2)$	$(1,0)$	$(1,3)''$	$(2,2)$	$(2,1)$	$(1,3)'$

We need the intertwining operators for the induced module from the Steinberg representation on A_2 in the Hecke algebra of type C_3, with parameters 2–2⇐1. The operators are

$$11 \times 1 : 1, \quad 1 \times 11 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}, \quad 111 \times 0 : \frac{\frac{3}{2} - \nu}{\frac{3}{2} + \nu}.$$

Therefore, the matching of operators is with the Hecke algebra of type G_2 with equal parameters, $\mathbb{H}(G_2)$. The nearby W-types match all the relevant $W(G_2)$-types, so the unitary set $\mathcal{U}(A_5'')$ is included in the spherical unitary dual of $\mathbb{H}(G_2)$. The modules of $\mathbb{H}(G_2)$ are parameterized by nilpotent orbits in the Lie algebra of type G_2. We analyze the composition series and unitarity of $X(A_5'')$ by cases corresponding to these nilpotent orbits.

(1) : In $\mathbb{H}(G_2)$, this is the spherical complementary series

$$\{3\nu_1 + 2\nu_2 < 1\} \cup \{3\nu_1 + \nu_2 > 1 > 2\nu_1 + \nu_2\}. \tag{5.3.1}$$

The standard module $X(A_5'')$ is irreducible in these regions, and therefore unitary (being unitarily induced and unitary when $\nu_2 = 0$).

A_1 : The parameters are of the form $(\nu_1, \nu_2) = (-\frac{1}{2} + \nu, 1)$. They are unitary for $0 \le \nu < \frac{1}{2}$, being endpoints of the complementary series (5.3.1). The decomposition of the standard module is

$$X(A_5'', (1, -\frac{1}{2} + \nu)) = \overline{X}(A_5'', (1, -\frac{1}{2} + \nu)) + X(A_5 + A_1, \nu)$$

$$\overline{X}(A_5'', (1, -\frac{1}{2} + \nu))|_W = 8_1 + 16_1 + 12_1 + 6_2 + 2 \cdot 8_4 + 4_3 + 4_5 + 2 \cdot 9_2$$
$$+ 9_4 + 2_4 + 1_2.$$

$\widetilde{A}_1$: The parameters are of the form $(\nu_1, \nu_2) = (1, -\frac{3}{2} + \nu, 1)$. They are unitary for $0 \le \nu < \frac{1}{2}$, being endpoints of the complementary series (5.3.1). The decomposition of the standard module is

$$X(A_5'', (-\frac{3}{2} + \nu, 1)) = \overline{X}(A_5'', (-\frac{3}{2} + \nu, 1)) + X(D_6(a_2), \nu)$$

$$\overline{X}(A_5'', (-\frac{3}{2} + \nu, 1))|_W = 8_1 + 6_1 + 16_1 + 12_1 + 2 \cdot 8_4 + 2 \cdot 4_3 + 2 \cdot 9_2$$
$$+ 9_4 + 2_4 + 1_2.$$

$G_2(a_1)$: This is the parameter $(\nu_1, \nu_2) = (0, 1)$. It is unitary, being an endpoint of the complementary series (5.3.1). The decomposition of the standard module is

$$X(A_5'', (0, 1)) = \overline{X}(A_5'', (0, 1)) + \overline{X}(A_5 A_1, \frac{1}{2}) + \overline{X}(D_6(a_2), \frac{1}{2})$$
$$+ X(E_7(a_5), \phi_1) + X(E_7(a_5), \phi_2)$$
$$\overline{X}(A_5'', (0, 1))|_W = 8_1 + 12_1 + 8_4 + 9_2 + 1_2.$$

G_2 : This is the parameter $(\nu_1, \nu_2) = (1, 1)$. In $\mathbb{H}(G_2)$, it is isolated. We compute the W-structure of the standard module. In addition to the operators given by the

nearby W-types, we need to check the operator on 9_2.

$$X(A_5'', (1,1)) = \overline{X}(A_5'', (1,1)) + \overline{X}(A_5 A_1, \frac{3}{2}) + \overline{X}(D_6(a_2), \frac{5}{2})$$

$$+ \overline{X}(D_5 A_1, \frac{1}{2}) + \overline{X}(D_6(a_1), \frac{1}{2}) + X(E_7(a_4), \phi_2)$$

$$\overline{X}(A_5'', (1,1))|_W = 8_1 + 9_2.$$

Moreover, the operator on 9_2 is positive at $(1,1)$, so this point is unitary.

In conclusion, the unitary parameter set $\mathcal{U}(A_5'')$ is identical with the spherical unitary dual $\mathcal{SU}(\mathbb{H}(G_2))$.

$D_4 + A_1$: The infinitesimal character is $(\nu_1, \nu_2, \frac{5}{2}, \frac{1}{2})$, the lowest W-type is 9_3, and $\mathfrak{z} = B_2$. In Langlands classification, the standard module is induced from the Steinberg representation on C_2. The W-structure is

$$X(D_4 + A_1)|_W = \mathrm{Ind}_{W(C_2)}^{W(F_4)}(0 \times 11).$$

The restrictions of the nearby W-types are as follows:

Nilpotent	$D_4 + A_1$	$D_5(a_1)A_1$	$D_6(a_2)$	$E_7(a_5)$	$D_6(a_1)$
W-type	9_3	4_1	16_1	$12_1, 6_2$	9_2
Multiplicity	1	1	2	$1, 1$	1
$B_2 \subset B_3$	1×11	0×12	$0 \times 12, 1 \times 11$	1×11	0×12
$C_2 \subset C_3$	0×12	0×12	$0 \times 12, 1 \times 11$	1×11	1×11
$W(\mathfrak{z}) = W(B_2)$	2×0	11×0	1×1	0×2	0×11

We need the intertwining operators for the induced modules from the Steinberg representation of $C_2 = B_2$ in the Hecke algebras of types B_3, and C_3, with parameters 1–1⇒2, respectively 2–2⇐1.

In B_3 (1–1⇒2), the infinitesimal character is $(\nu, 3, 2)$, and the standard module is the induced from the Steinberg representation on B_2. The operators are

$$1 \times 11 : 1, \quad 0 \times 12 : \frac{1 - \nu}{1 + \nu}, \quad 0 \times 111 : \frac{4 - \nu}{4 + \nu}.$$

In C_3 (2–2⇐1), the infinitesimal character is $(\nu, \frac{5}{2}, \frac{1}{2})$, and the standard module is the induced from the Steinberg representation on C_2. The operators are

$$0 \times 12 : 1, \quad 1 \times 11 : \frac{\frac{3}{2} - \nu}{\frac{3}{2} + \nu}, \quad 0 \times 111 : \frac{(\frac{3}{2} - \nu)(\frac{9}{2} - \nu)}{(\frac{3}{2} + \nu)(\frac{9}{2} + \nu)}.$$

Therefore, the matching of operators is with the Hecke algebra of type B_2 with parameters 1⇒3/2. The nearby W-types match all the $W(B_2)$-representations of this B_2, therefore, the unitary parameter set $\mathcal{U}(D_4 + A_1)$ is included in the spherical unitary dual of $\mathbb{H}(B_2, 1, 3/2)$. The modules of $\mathbb{H}(B_2, 1, 3/2)$ are parameterized by a cuspidal local system on $\mathfrak{sp}(2) \oplus \mathbb{C}^2 \subset \mathfrak{sp}(6)$. Specifically the nilpotent orbits of $\mathfrak{sp}(6)$ which enter in the parameterization are $(6), (42), (411), (222), (21^4)$. We analyze the composition series and unitarity of $X(D_4 + A_1)$ by cases corresponding to these nilpotent orbits.

(21^4) : In $H(B_2, 1, 3/2)$, this is the spherical complementary series

$$\{\nu_1 + \nu_2 < 1\} \cup \{\nu_1 - \nu_2 > 1, \nu_1 < \frac{3}{2}\}. \tag{5.3.2}$$

The standard module $X(D_4 + A_1)$ is irreducible in these regions, and therefore unitary (being unitarily induced and unitary when $\nu_2 = 0$).

(222) : The parameters are of the form $(\nu_1, \nu_2) = (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)$. They are unitary for $0 \le \nu < \frac{3}{2}$, being endpoints of the complementary series (5.3.2). The standard module decomposes as follows

$$X(D_4 A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)) = \overline{X}(D_4 A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu) + X(D_5(a_1)A_1, \nu)$$

$$\overline{X}(D_4 A_1, (\frac{1}{2} + \nu, -\frac{1}{2} + \nu)|_W = 9_3 + 16_1 + 12_1 + 2 \cdot 9_4 + 8_4 + 2 \cdot 8_2 + 6_2$$
$$+ 2 \cdot 4_5 + 2_2 + 1_4.$$

(411) : The parameters are $(\nu_1, \nu_2) = (\nu, \frac{3}{2})$. They are unitary for $0 \le \nu \le \frac{1}{2}$, being endpoints of the complementary series (5.3.2). The standard module decomposes as follows

$$X(D_4 A_1, (\nu, \frac{3}{2})) = \overline{X}(D_4 A_1, (\nu, \frac{3}{2})) + X(D_6(a_2), \nu)$$

$$\overline{X}(D_4 A_1, (\nu, \frac{3}{2}))|_W = 9_3 + 16_1 + 8_2 + 4_1 + 2_2 + 9_4.$$

(42) : The parameters are $(\nu_1, \nu_2) = (\frac{3}{2}, \frac{1}{2})$. This is an endpoint of the complementary series (5.3.2), therefore unitary. The decomposition of the standard module is

$$X(D_4 A_1, (\frac{3}{2}, \frac{1}{2})) = \overline{X}(D_4 A_1, (\frac{3}{2}, \frac{1}{2})) + \overline{X}(D_5(a_1)A_1, 1) + \overline{X}(D_6(a_2), \frac{1}{2})$$
$$+ X(E_7(a_5), \phi_1) + X(E_7(a_5), \phi_2)$$
$$\overline{X}(D_4 A_1, (\frac{3}{2}, \frac{1}{2}))|_W = 9_3 + 16_1 + 8_2 + 9_4 + 2_2.$$

(6) : The parameters are $(\nu_1, \nu_2) = (\frac{5}{2}, \frac{3}{2})$. This point is isolated. From the W-structure of $\overline{X}(D_4 A_1, (\nu, \frac{3}{2}))$, we see that the only W-types which can be present in $\overline{X}(D_4 A_1, (\frac{5}{2}, \frac{3}{2}))$ are $9_3, 16_1, 8_2, 4_1, 2_2, 9_4$. By virtue of the matching with operators in $\mathbb{H}(B_2, 1, 3/2)$, we know 16_1 and 4_1 cannot be present. We compute the operators on the remaining ones and find that the W-structure is

$$\overline{X}(D_4 A_1, (\frac{5}{2}, \frac{3}{2})) = 9_3 + 8_2.$$

Moreover, the operator on 8_2 is positive, so $\overline{X}(D_4 A_1, (\frac{5}{2}, \frac{3}{2}))$ is unitary. Note also that $\overline{X}(D_4 A_1, (\frac{5}{2}, \frac{3}{2}))$ is the IM-dual of $\overline{X}(A_5'', (1, 1))$.

In conclusion, the unitary set $\mathcal{U}(D_4 + A_1)$ is identical with $\mathcal{SU}(\mathbb{H}(B_2, 1, 3/2)$.

$\mathbf{D_4(a_1) + A_1}$: The infinitesimal character is $(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$, the lowest W-types are 9_1 and 2_1, and $\mathfrak{z} = 2A_1$. The two lowest W-types are separate when $\nu_1 = \nu_2$. If $\nu_1 \neq \nu_2$, then the (Langlands classification) standard module is induced from a one dimensional discrete series on B_2 (1⇒2). As a W-module

$$X(D_4(a_1)A_1, (\nu_1, \nu_2))|_W = \mathrm{Ind}_{W(B_2)}^{W(F_4)}(0 \times 2), \quad \text{if } \nu_1 \neq \nu_2.$$

When $\nu_1 = \nu_2 = \nu$, there are two standard modules, corresponding to the two lowest W-types, induced from two tempered representations of B_3 (1–1⇒2). As

W-modules,

$$X(D_4(a_1)A_1, (\nu,\nu), \phi_1)|_W = \mathrm{Ind}_{W(B_3)}^{W(F_4)}(1 \times 2 + 0 \times 12) = 9_1 + 8_1 + 4_3 + 6_1$$
$$+ 2 \cdot 9_2 + 2 \cdot 16_1 + 2 \cdot 8_4 + 9_4 + 12_1 + 4_1 + 2_4.$$

$$X(D_4(a_1)A_1, (\nu,\nu), \phi_2)|_W = \mathrm{Ind}_{W(B_3)}^{W(F_4)}(0 \times 3) = 2_1 + 8_1 + 4_3 + 9_2 + 1_2.$$

The restrictions of the nearby W-types are:

Nilpotent	$D_4(a_1)A_1$	$D_4(a_1)A_1$	A_5''	$A_3A_2A_1$	$D_5(a_1)A_1$
W-type	9_1	2_1	8_1	4_3	4_1
Multiplicity	$1+0$	$0+1$	$1+1$	$1+1$	$1+0$
$B_2 \subset B_3$	1×2	0×3	$1 \times 2, 0 \times 3$	$1 \times 2, 0 \times 3$	0×12
$C_2 \subset C_3$	12×0	12×0	$12 \times 0, 11 \times 1$	$11 \times 1, 111 \times 0$	0×12
$W(B_2) =$	2×0	11×0	1×1	1×1	11×0
$W(\tilde{\mathfrak{z}}) \rtimes \mathbb{Z}/2$					

The matching of the intertwining operators is with the Hecke algebra $\mathbb{H}(B_2, 0, 5/2)$. More specifically,

$9_1 :$ 1;

$2_1 :$ 1;

$8_1 :$ $\begin{pmatrix} \frac{\frac{5}{2}-\nu_1}{\frac{5}{2}-\nu_2} & 0 \\ 0 & \frac{\frac{5}{2}-\nu_2}{\frac{5}{2}+\nu_2} \end{pmatrix}.$

For the calculations, we need some intertwining operators on modules in the Hecke algebras of type B_3 and C_3.

In $\mathbb{H}(B_3, 1, 2)$, the infinitesimal character is $(\nu, 2, 1)$. There are two lowest W-types 1×2 and 0×3, which are separate at $\nu = 0$ only. When $\nu > 0$, the operators are

$$1 \times 2 : 1, \quad 0 \times 3 : 1, \quad 0 \times 12 : \frac{3-\nu}{3+\nu}.$$

In $\mathbb{H}(C_3, 2, 1)$, the infinitesimal character is $(\nu, \frac{3}{2}, \frac{1}{2})$. The lowest W-type is 12×0. The operators are

$$12 \times 0 : 1, \quad 11 \times 1 : \frac{\frac{5}{2}-\nu}{\frac{5}{2}+\nu}, \quad 111 \times 0 : \frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}.$$

From this calculations, it follows that the reducibility lines for $X(D_4(a_1), A_1, (\nu_1, \nu_2))$, when $\nu_1 \neq \nu_2$, are $\nu_1 \pm \nu_2 = 3$, $\nu_i = \frac{5}{2}, \frac{7}{2}$, $i = 1, 2$. The operator on 8_1 shows that the unitary set $\mathcal{U}(D_4(a_1)A_1)$ is included in $0 \leq \nu_2 \leq \nu \leq \frac{5}{2}$. However, the line $\nu_1 + \nu_2 = 3$ cuts this region. The restrictions in the table above imply that the operator on 4_1 is

$$4_1 : \frac{(3-(\nu_1+\nu_2))(3-(\nu_1-\nu_2))}{(3+(\nu_1+\nu_2))(3+(\nu_1-\nu_2))}.$$

So for $\nu_1 \neq \nu_2$, $X(D_4(a_1)A_1, (\nu_1, \nu_2))$ is unitary if and only if

$$\{\nu_1 + \nu_2 < 3, \ \nu_1 < \frac{5}{2}\}.$$

We also note that $\overline{X}(D_4(a_1)A_1, (\nu, \frac{5}{2})), 0 \leq \nu < \frac{1}{2}$ is the IM-dual of $\overline{X}(D_4A_1, (\nu, \frac{3}{2}))$, and $\overline{X}(D_4(a_1)A_1, (\frac{5}{2}, \frac{1}{2}))$ is the IM-dual of $\overline{X}(D_4A_1, (\frac{3}{2}, \frac{1}{2}))$. On $\nu_1 + \nu_2 = 3$, we write the parameters as $(\frac{3}{2}+\nu, -\frac{3}{2}+\nu)$ (which are unitary for $0 \leq \nu \leq 1$). The

W-structure here is

$$X(D_4(a_1)A_1, (\frac{3}{2}+\nu, -\frac{3}{2}+\nu)) = \overline{X}(D_4(a_1)A_1, (\frac{3}{2}+\nu, -\frac{3}{2}+\nu)) + X(D_5(a_1)A_1, \nu)$$

$$\overline{X}(D_4(a_1)A_1, (\frac{3}{2}+\nu, -\frac{3}{2}+\nu))|_W = 9_1 + 2_1 + 2 \cdot 9_4 + 16_1 + 8_4 + 2 \cdot 8_1$$
$$+ 2 \cdot 4_3 + 12_1 + 6_1 + 1_2.$$

It remains to analyze the case $\nu_1 = \nu_2 = \nu$. The calculation on 8_1 can be used to conclude that both Langlands quotients $\overline{X}(D_4(a_1)A_1, (\nu, \nu))$ fail to be unitary for $\nu > \frac{5}{2}$. Moreover, the calculation for 4_1 implies that $\overline{X}(D_4(a_1)A_1, (\nu, \nu), \phi_1)$ is not unitary for $\nu > \frac{3}{2}$.

For ϕ_1, the first possible reducibility point is $\nu = \frac{3}{2}$ (this is seen by looking at the W-types in $X(D_4(a_1)A_1, (\nu, \nu), \phi_1)$ and their corresponding nilpotent orbits). So it is unitary for $0 \leq \nu \leq \frac{3}{2}$. The decomposition at $\nu = \frac{3}{2}$ is

$$X(D_4(a_1)A_1, (\frac{3}{2}, \frac{3}{2}), \phi_1) = \overline{X}(D_4(a_1)A_1, (\frac{3}{2}, \frac{3}{2}), \phi_1) + X(D_5(a_1)A_1, 0)$$

$$X(D_4(a_1)A_1, (\frac{3}{2}, \frac{3}{2}), \phi_1)|_W = 9_1 + 9_2 + 16_1 + 8_4 + 8_1 + 12_1 + 6_1 + 4_3.$$

For ϕ_2, the first possible reducibility point is $\nu = \frac{5}{2}$, so this is unitary if and only if $0 \leq \nu \leq \frac{5}{2}$. Note that $\overline{X}(D_4(a_1)A_1, (\frac{5}{2}, \frac{5}{2}), \phi_2)$ is the IM-dual of $X(E_7(a_4), \phi_1)$. As a W-representation, it is just 2_1.

In conclusion, the unitary set $\mathcal{U}(D_4(a_1)A_1)$ is given by:
(1) $\{\nu_1 + \nu_2 \leq 3, \; \nu_1 \leq \frac{5}{2}\}$, if $\nu_1 \neq \nu_2$.
(2) $0 \leq \nu \leq \frac{3}{2}$, if $\nu_1 = \nu_2 = \nu$ for the ϕ_1-factor.
(3) $0 \leq \nu \leq \frac{5}{2}$, if $\nu_1 = \nu_2 = \nu$ for the ϕ_2-factor.

$\mathbf{A_3 + 2A_1}$: The infinitesimal character is $(\nu_1, 1 + \nu_2, -1 + \nu_2, \frac{1}{2})$, the lowest W-type 8_3, and $\mathfrak{z} = 2A_1$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on the Hecke algebra of type $A_1 + \widetilde{A}_1$. As a W-module,

$$X(A_3 + 2A_1)|_W = \mathrm{Ind}_{W(A_1 + \widetilde{A}_1)}^{W(F_4)}((11) \otimes (11)).$$

The restrictions of nearby W-types are:

Nilpotent	$A_3 + 2A_1$	$D_4(a_1)A_1$	$A_3A_2A_1$	A_5''	D_4A_1	$D_5(a_1)A_1$
W-type	8_3	9_1	4_4	8_1	9_3	4_1
Multiplicity	1	1	1	1	2	1
$C_1 \times A_1 \subset C_3$	1×2	1×2	1×2	11×1	$1 \times 2, 0 \times 12$	0×12
$B_1 \times A_1 \subset B_3$	11×1	1×2	11×1	1×2	$1 \times 2, 2 \times 1$	0×12
$A_1 \subset A_2$	(21)	(21)	(1^3)	(21)	$(21), (1^3)$	(21)
$\widetilde{A}_1 \subset \widetilde{A}_2$	(21)	(21)	(1^3)	(21)	$(21), (21)$	(21)

We need the following calculations from the Hecke algebras of type B_3 and C_3.

In $\mathbb{H}(C_3, 2, 1)$, the infinitesimal character is $(1 + \nu, -1 + \nu, \frac{1}{2})$. The standard module is induced from the Steinberg module on $C_1 \times A_1$, and it is reducible at $\nu = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$. The relevant operators are:

$$1 \times 2 : 1, \quad 11 \times 1 : \frac{\frac{5}{2} - \nu}{\frac{5}{2} + \nu}, \quad 0 \times 12 : \frac{\frac{3}{2} - \nu}{\frac{3}{2} + \nu}.$$

In $\mathbb{H}(B_3, 1, 2)$, the infinitesimal character is $(\frac{1}{2} + \nu, -\frac{1}{2} + \nu, 2)$. The standard module is induced from the Steinberg module on $B_1 \times A_1$, and it is reducible at $\nu = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}$. The relevant operators are:

$$11 \times 1 : 1, \quad 1 \times 2 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}, \quad 1 \times 11 : \frac{\frac{5}{2} - \nu}{\frac{5}{2} + \nu}.$$

The reducibility lines for $X(A_3 + 2A_1, (\nu_1, \nu_2))$ are $\nu_1 = \frac{1}{2}, \frac{5}{2}, \frac{7}{2}$, $\nu_2 = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, and $\pm\nu_1 \pm 2\nu_2 = \frac{3}{2}$ and $\pm\nu_1 \pm \nu_2 = 3$. (The last two types come from the restrictions of the operators to $A_1 \subset A_2$ and $\widetilde{A}_1 \subset \widetilde{A}_2$.)

From the restrictions we that the operators are as follows:

$8_3 :$ 1;

$9_1 :$ $\frac{\frac{1}{2} - \nu_1}{\frac{1}{2} + \nu_1}$;

$4_1 :$ $\frac{(\frac{1}{2} - \nu_1)(\frac{5}{2} - \nu_1)(\frac{3}{2} - \nu_2)}{(\frac{1}{2} + \nu_1)(\frac{5}{2} + \nu_1)(\frac{3}{2} + \nu_2)}$;

$8_1 :$ $\frac{(\frac{1}{2} - \nu_1)(\frac{5}{2} - \nu_2)}{(\frac{1}{2} + \nu_1)(\frac{5}{2} + \nu_2)}$;

$4_4 :$ $\frac{(\frac{3}{2} - (\nu_1 + 2\nu_2))(\frac{3}{2} - (\nu_1 - 2\nu_2))}{(\frac{3}{2} + (\nu_1 + 2\nu_2))(\frac{3}{2} + (\nu_1 - 2\nu_2))}$;

$9_3 :$ $\text{Det} = \frac{(\frac{3}{2} - \nu_2)(\frac{5}{2} - \nu_1)(\frac{3}{2} - (\nu_1 + 2\nu_2))(\frac{3}{2} - (\nu_1 - 2\nu_2))}{(\frac{3}{2} + \nu_2)(\frac{5}{2} + \nu_1)(\frac{3}{2} + (\nu_1 + 2\nu_2))(\frac{3}{2} + (\nu_1 - 2\nu_2))}$.

In this case, it is natural to try to match the unitary set $\mathcal{U}(A_3 + 2A_1)$ with the spherical unitary dual of $A_1 + A_1$ with parameters 1 and 3. This is just the set $\{0 \le \nu_1 \le \frac{1}{2},\ 0 \le \nu_2 \le \frac{3}{2}\}$. The operators above imply that actually the unitary set is

$$\mathcal{U}(A_3 + 2A_1) = \{0 \le \nu_1 \le \frac{1}{2}, \nu_1 + 2\nu_2 \le \frac{3}{2}\} \cup \{0 \le \nu_1 \le \frac{1}{2}, 0 \le \nu_2 \le \frac{3}{2}, 2\nu_2 - \nu_1 \ge \frac{3}{2}\}.$$

We also record the relevant decompositions and W-structure of the standard module. On $\nu_2 = \frac{3}{2}$, $0 \le \nu_1 < \frac{1}{2}$, the decomposition is

$$X(A_3 + 2A_1, (\nu, \frac{3}{2})) = \overline{X}(A_3 + 2A_1, (\nu, \frac{3}{2})) + X(D_4 + A_1, (\nu, \frac{1}{2}).$$

Moreover, $\overline{X}(A_3 + 2A_1, (\nu, \frac{3}{2}))$ is self IM-dual.

On $\nu_1 = \frac{1}{2}$, $0 \le \nu_2 < \frac{1}{2}$ and $1 < \nu_2 < \frac{3}{2}$, the decomposition is

$$X(A_3 + 2A_1, (\frac{1}{2}, \nu)) = \overline{X}(A_3 + 2A_1, (\frac{1}{2}, \nu)) + X(D_4(a_1) + A_1, (\nu, \nu), \phi_1).$$

The IM-dual of $\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \nu))$ is a spherical module.

On $2\nu_2 \pm \nu_1 = \frac{3}{2}$, we write the parameter as $(\nu_1, \nu_2) = (-\frac{1}{2} + 2\nu, \frac{1}{2} + \nu)$. Then it is unitary for $0 < \nu < \frac{1}{2}$. Here, the decomposition is

$$X(A_3 + 2A_1, (-\frac{1}{2} + 2\nu, \frac{1}{2} + \nu)) = \overline{X}(A_3 + 2A_1, (-\frac{1}{2} + 2\nu, \frac{1}{2} + \nu)) + X(A_3 + A_2 + A_1, \nu).$$
$$(5.3.3)$$

The IM-dual of $\overline{X}(A_3 + 2A_1, (-\frac{1}{2} + 2\nu, \frac{1}{2} + \nu))$ is parameterized by $4A_1$.

At $(\frac{1}{2}, \frac{1}{2})$, the decomposition is

$$X(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2})) = \overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2})) + X(D_4(a_1)A_1, (\frac{1}{2}, \frac{1}{2}), \phi_1) + X(A_3 A_2 A_1, 0)$$

$$\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2}))|_W = 8_3 + 16_1 + 8_2 + 12_1 + 6_2 + 9_3 + 9_4 + 4_5.$$

The similar decomposition (and identical W-structure) holds at $(1, 1)$.

Finally, at the point $(\frac{1}{2}, \frac{3}{2})$, the decomposition is

$$X(A_3 + 2A_1, (\frac{1}{2}, \frac{3}{2})) = \overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{3}{2})) + \overline{X}(D_4(a_1)A_1, (\frac{3}{2}, \frac{3}{2}), \phi_1)$$

$$+ \overline{X}(D_4 A_1, (\frac{1}{2}, \frac{1}{2})) + X(D_5(a_1)A_1, 0)$$

$$\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{3}{2}))|_W = 8_3 + 16_1 + 8_2 + 4_4 + 12_1 + 6_1 + 9_3 + 9_4.$$

Note that $\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{3}{2}))$ is the IM-dual of $\overline{X}(D_4(a_1)A_1, (\frac{3}{2}, \frac{3}{2}), \phi_1)$.

$(\mathbf{A_3 + A_1})''$: The infinitesimal character is $(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1 - \nu_2}{2}, 1 + \frac{\nu_2}{2}, -1 + \frac{\nu_2}{2})$, the lowest W-type is 4_2, and centralizer B_3. In the Langlands classification, the standard module is induced from the Steinberg representation on $\widetilde{A}_1$, so as a W-module, it is $X((A_3 + A_1)'')|_W = \text{Ind}_{W(\widetilde{A}_1)}^{W(F_4)}((11))$.

The restrictions of nearby W-types are:

Nilpotent	$(A_3 A_1)''$	$A_3 2A_1$	$D4(a_1)A_1$	$D_4(a_1)A_1$	A_5''
W-types	4_2	8_3	9_1	2_1	8_1
$\widetilde{A}_1 \subset \widetilde{A}_2$	(21)	$2 \cdot (21)$	$3 \cdot (21)$	(21)	$3 \cdot (21), (1^3)$
A_1	(2)	$(2), (11)$	$2 \cdot (2), (11)$	(2)	$3 \cdot (2), (11)$
$A_1 \subset C_2$	1×1	$2(1 \times 1)$	$2(1 \times 1), 11 \times 0$	11×0	$2(1 \times 1), 2(11 \times 0)$
$W(\mathfrak{z})$	3×0	12×0	2×1	0×3	$3 \times 0 + 1 \times 2$

We need a calculation from the Hecke algebra of type C_2, $2 \Leftarrow 1$. The infinitesimal character is $(1 + \nu, -1 + \nu)$, and the standard module is induced from the Steinberg representation on $\widetilde{A}_1$. Then the reducibility points are $\nu = \frac{1}{2}, \frac{3}{2}$, and the operators are

$$1 \times 1 : 1, \quad 11 \times 0 : \frac{\frac{1}{2} - \nu}{\frac{1}{2} + \nu}.$$

It follows that the reducibility hyperplanes for $X((A_3 + A_1)'', (\nu_1, \nu_2))$ are $\nu_i = 1, 3$ (from $A_1 \subset C_2$), $\nu_i \pm \nu_j = 1$ (from $\widetilde{A}_1 \subset \widetilde{A}_1 + A_1$), and $\nu_1 \pm \nu_2 \pm \nu_3 = 6$ (from $\widetilde{A}_1 \subset \widetilde{A}_2$).

The matching of operators is with the spherical operators for $\mathbb{H}(B_3)$, the Hecke algebra of type B_3 with equal parameters. From the tables, we see that we can match the operators on all relevant $W(B_3)$-types, except 1×2. (The operator on 8_1 fails to match it because of the restriction to $\widetilde{A}_1 \subset \widetilde{A}_2$.) The representation 1×2 in B_3 is the only one which rules out the interval $\{(1 + \nu, \nu, -1 + \nu) : 1 < \nu < 2\}$ in the spherical dual. It follows that the unitary set $\mathcal{U}((A_3 A_1)'')$ is included in the union $\mathcal{SU}(\mathbb{H}(B_3)) \cup \{(1 + \nu, \nu, -1 + \nu) : 1 < \nu < 2\}$. We also note that the extra hyperplanes of reducibility of $X((A_3 + A_1)'')$ which do not correspond to B_3, intersect this union only in the point $(3, 2, 1)$. We analyze the parameters in $\mathcal{U}((A_3 A_1)'')$ partitioned by the nilpotent orbits in type $\mathfrak{so}(7)$ (that is, in the same way we parameterize the spherical dual of $\mathbb{H}(B_3)$).

(1^7) : The parameters are (ν_1, ν_2, ν_3); the spherical complementary series for $\mathbb{H}(B_3)$ is

$$\{\nu_1 + \nu_2 < 1\} \cup \{\nu_1 + \nu_3 > 1, \nu_2 + \nu_3 < 1, \nu_1 < 1\}. \tag{5.3.4}$$

In these regions, $X((A_3 + A_1)'')$ is irreducible, and it is unitarily induced and unitary for $\nu_3 = 0$, so it is unitary in (5.3.4).

(221^3) : The parameters are $(\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2)$, unitary for $\{0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < \frac{1}{2}\}$, being endpoints of the complementary series (5.3.4). The decomposition

of the standard module in this region is

$$X((A_3A_1)'', \frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, 2\nu_2)) = \overline{X}((A_3A_1)'', \frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, 2\nu_2))$$
$$+ X(A_3+2A_1, (\nu_1, \nu_2)).$$

The IM-dual of $\overline{X}((A_3A_1)'', \frac{1}{2}+\nu_1, -\frac{1}{2}+\nu_1, 2\nu_2))$ is parameterized by $4A_1$.

(31^4) : The parameters are $(\nu_1+\nu_2, \nu_1-\nu_2, 1)$, unitary for $\{\nu_1 < \frac{1}{2}\}$, being endpoints of (5.3.4). The decomposition of the standard module is

$$X((A_3A_1)'', (\nu_1+\nu_2, \nu_1-\nu_2, 1)) = \overline{X}((A_3A_1)'', (\nu_1+\nu_2, \nu_1-\nu_2, 1))$$
$$+ X(D_4(a_1)A_1, (\nu_1, \nu_2)).$$

The IM-dual of $\overline{X}((A_3A_1)'', (\nu_1+\nu_2, \nu_1-\nu_2, 1))$ is a spherical module.

(322) : The parameters are $(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, 1)$, unitary for $0 \le \nu < \frac{1}{2}$ (again endpoints of (5.3.4). The standard module decomposes as

$$X((A_3A_1)'', (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, 1)) = \overline{X}((A_3A_1)'', (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, 1))$$
$$+ X(A_3+2A_1, (\nu, \frac{1}{2}) + X(D_4(a_1)A_1, (\nu, \frac{1}{2}))$$

Moreover, $\overline{X}((A_3A_1)'', (\frac{1}{2}+\nu, -\frac{1}{2}+\nu, 1))$ is self IM-dual.

(331) : The parameters are $(1+\nu, \nu, -1+\nu)$. At $\nu = 0$, we have a unitary module, endpoint of (5.3.4). (Note also that $\overline{X}((A_3A_1)'', (1,1,0))$ is the IM-dual of $\overline{X}(A_3+2A_1, (\frac{1}{2}, \frac{1}{2}))$.) In $\mathbb{H}(B_3)$, this is the only such parameter which is unitary. In F_4 however, by the observations preceding this analysis, we need to also consider the segment $1 < \nu < 2$. In this interval, the decomposition is

$$X((A_3+A_1)'', (1+\nu, \nu, -1+\nu)) = \overline{X}((A_3+A_1)'', (1+\nu, \nu, -1+\nu))$$
$$+ 2 \cdot \overline{X}(A_3+2A_1, (-\frac{1}{2}+\nu, \frac{1}{2}+\frac{\nu}{2})) + X(A_3A_2A_1, \frac{\nu}{2})$$
$$\overline{X}((A_3+A_1)'', (1+\nu, \nu, -1+\nu))|_W = 4_2 + 9_1 + 2_1 + 6_2 + 2 \cdot 8_1 + 4_3$$
$$+ 2 \cdot 9_2 + 12_1 + 16_1 + 8_4 + 1_2.$$

This segment is isolated, so to prove that it is unitary, we compute explicitly the signatures on all the W-types which appear in the restriction of $\overline{X}((A_3+A_1)'', (1+\nu, \nu, -1+\nu))|_W$. The IM-dual of $\overline{X}((A_3+A_1)'', (1+\nu, \nu, -1+\nu))$ is a module parameterized by A_2+3A_1, and our calculations will be confirmed by the unitarity of those duals.

(511) : The parameters are $(\nu, 2, 1)$, which are not unitary for $\nu > 0$. At $\nu = 0$, $\overline{X}((A_3A_1)'', (2,1,0))$ is the IM-dual of $\overline{X}(A_3+2A_1, (\frac{1}{2}, 1))$, and therefore it is unitary.

(7) : The parameter is $(3, 2, 1)$. Then $\overline{X}((A_3+A_1)'', (3,2,1))$ is the IM-dual of $X(E_7(a_5), \phi_2)$, and therefore it is unitary.

In conclusion, $\mathcal{U}((A_3+A_1)'') = \mathcal{SU}(\mathbb{H}(B_3)) \sqcup \{(1+\nu, \nu, -1+\nu) : 1 < \nu < 2\}$.

$\mathbf{A_2+3A_1}$: The infinitesimal character is $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu_1(2, 1, 1, 0) + \nu_2(1, 1, 0, 0)$, the lowest W-type is 1_3, and $\mathfrak{z} = G_2$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on A_2, and as a W-module, $X(A_2+3A_1) = \mathrm{Ind}_{W(A_2)}^{W(F_4)}((111))$.

The restrictions of nearby W-types are:

Nilpotent	$A_2 3A_1$	$A_3 2A_1$	$A_3 A_2 A_1$	$D_4 A_1$	$A_5 A_1$
W-type	1_3	8_3	4_4	9_3	6_1
Multiplicity	1	1	2	3	1
$A_2 \subset B_3$	111×0	11×1	111×0	$111 \times 0, 1 \times 11$	11×1
			11×1	11×1	
$A_2 \subset A_2 + \widetilde{A}_1$	(2)	(2)	$(2), (11)$	$2 \cdot (2), (11)$	(11)
$W(\mathfrak{z})$	$(1,0)$	$(1,3)'$	$(2,1)$	$(1,0) + (2,2)$	$(1,6)$

We need a calculation in the Hecke algebra $\mathbb{H}(B_3, 1, 2)$. The infinitesimal character is $(\nu + 1, \nu, -1 + \nu)$, and the standard module is induced from the Steinberg representation on A_2. The reducibility points are $1, 2, 3$, and the relevant operators are

$$111 \times 0 : 1, \quad 11 \times 1 : \frac{1 - \nu}{1 + \nu}, \quad 1 \times 11 : \frac{(1 - \nu)(2 - \nu)}{(1 + \nu)(2 + \nu)}.$$

Then the reducibility lines for $X(A_2 + 3A_1, (\nu_1, \nu_2))$ are $\nu_1, \nu_1 + \nu_2, 2\nu_1 + \nu_2 = 1, 2, 3$, and $\nu_2, 3\nu_1 + \nu_2, 3\nu_1 + 2\nu_2 = 2$. The matching of operators is with $\mathbb{H}(G_2, 1, 2)$, the Hecke algebra of type G_2: $1 < \equiv 2$. From the tables, it follows that we are able to match only the operators on $(1, 0)$, $(1, 3)'$, and $(2, 1)$ in G_2. We will also use the operator on 9_3 (which is the closest one to match $(2, 2)$). The spherical unitary dual of $\mathbb{H}(G_2, 1, 2)$ consists of the closed region $\{2\nu_1 + \nu_2 \leq 1\}$, and the isolated points $(\frac{1}{2}, \frac{1}{2})$ and $(1, 2)$. The matched operators are sufficient to conclude that, when irreducible, $X(A_2 + 3A_1, (\nu_1, \nu_2))$ is unitary only in the region $2\nu_1 + \nu_2 < 1$, but on the lines we need more information.

On the line $\nu_1 = -1 + \nu, \nu_2 = 2$, the decomposition is

$$X(A_2 + 3A_1, (-1 + \nu, 2)) = \overline{X}(A_2 + 3A_1, (-1 + \nu, 2)) + X(A_3 + A_2 + A_1, \nu)$$

$$\overline{X}(A_2 + 3A_1, (-1 + \nu, 2))|_W = 1_3 + 8_3 + 4_4 + 6_2 + 16_1 + 2 \cdot 8_2 + 2 \cdot 9_3$$
$$+ 12_1 + 9_4 + 2_2 + 4_5.$$

To simplify notation, set $X_1(\nu) = X(A_2 + 3A_1, (-1 + \nu, 2))$. The matched operators give

$$8_3 : -\frac{(\frac{1}{2} - \nu)(2 - \nu)}{(\frac{1}{2} + \nu)(2 + \nu)}, \quad 4_4 : \frac{2 - \nu}{2 + \nu},$$

showing that $X_1(\nu)$ can only be unitary for $\frac{1}{2} \leq \nu \leq 2$. The operator on 9_3 has two nonzero eigenvalues with product $-\frac{(\frac{1}{2} - \nu)(1 - \nu)(2 - \nu)^2(3 - \nu)}{(\frac{1}{2} + \nu)(1 + \nu)(2 + \nu)^2(3 + \nu)}$, so the only segment which can be unitary is $\frac{1}{2} \leq \nu \leq 1$. At $\nu = \frac{1}{2}$, $X_1(\nu)$ decomposes further:

$$X_1(\frac{1}{2}) = \overline{X}(A_2 + 3A_1, (\frac{1}{2}, \frac{1}{2})) + \overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2}))$$

$$\overline{X}(A_2 + 3A_1, (\frac{1}{2}, \frac{1}{2}))|_W = 1_3 + 4_4 + 8_2 + 9_3 + 2_2.$$

Since $X(A_2 + 3A_1, (\frac{1}{2}, \frac{1}{2}))$ is the IM-dual of $X(D_4(a_1)A_1, \phi_2)$ (which is tempered), it must be unitary. Also the other factor is unitary, because $\overline{X}(A_3 + 2A_1, (\frac{1}{2}, \frac{1}{2}))$ is. Since 9_2 appears in both factors, and the operator on it is positive for $\frac{1}{2} < \nu < 1$, it follows that $X_1(\nu)$ is unitary for $\frac{1}{2} < \nu < 1$. Note also that $X_1(\nu)$ is the IM-dual of $\overline{X}((A_3 + A_1)'', 1 + 2\nu, 2\nu, -1 + 2\nu)$, which confirms its unitarity.

On the line $\nu_1 = 1, \nu_2 = -\frac{3}{2} + \nu$, the decomposition is

$$X(A_2 + 3A_1, (1, -\frac{3}{2} + \nu)) = \overline{X}(A_2 + 3A_1, (1, -\frac{3}{2} + \nu)) + \overline{X}(A_3 + 2A_1, (\frac{1}{2}, \nu))$$

$$\overline{X}(A_2 + 3A_1, (1, -\frac{3}{2} + \nu))|_W = 1_3 + 4_4 + 8_2 + 9_3 + 2_2. \tag{5.3.5}$$

Denote $X_2(\nu) = \overline{X}(A_2 + 3A_1, (1, -\frac{3}{2} + \nu))$. The intertwining operators calculations give us

$$4_4 : \frac{\frac{7}{2} - \nu}{\frac{7}{2} + \nu}, \quad 9_3 : \frac{(\frac{5}{2} - \nu)(\frac{7}{2} - \nu)}{(\frac{5}{2} + \nu)(\frac{7}{2} + \nu)}.$$

So the only interval on which $X_2(\nu)$ can be unitary is $0 \le \nu \le \frac{5}{2}$. But on this interval, $X_2(\nu)$ is the IM-dual of $X(D_4(a_1)A_1, (\nu, \nu), \phi_2)$, so it is in fact unitary. Note that at the endpoint, $\nu = \frac{5}{2}$, $\overline{X}(A_2 + 3A_1, (1, 1))$ is the IM-dual of the discrete series $X(E_7(a_4), \phi_2)$.

Finally, at the point $(1, 2)$, $\overline{X}(A_2 + 3A_1, (1, 2))$ is the IM-dual of the discrete series $X(E_7(a_3), \phi_2)$.

In conclusion, the unitary set $\mathcal{U}(A_2 + 3A_1)$ is the one pictured in the figure 3. It is strictly larger than the spherical unitary dual $\mathcal{SU}(\mathbb{H}(G_2, 1, 2))$.
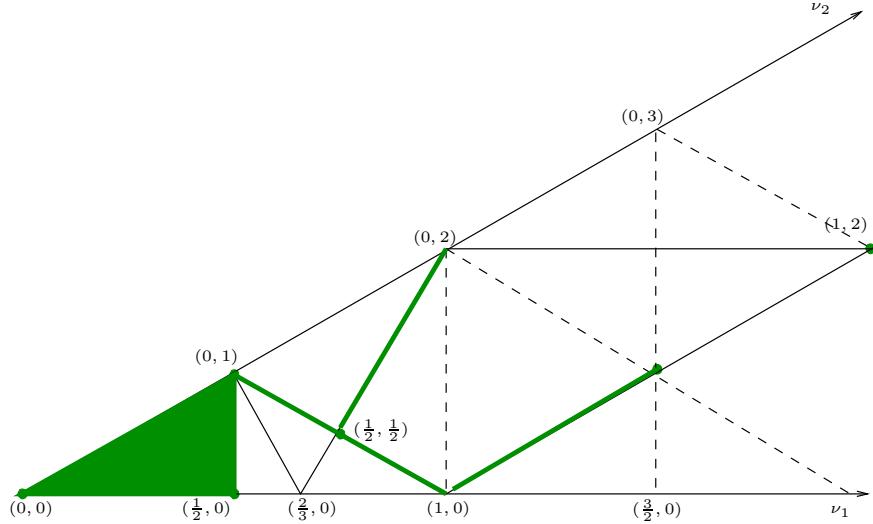
FIGURE 3. Unitary parameters and reducibility lines for $A_2 + 3A_1$

$\mathbf{4A_1}$: The parameters are $(\nu_1, \nu_2, \nu_3, \frac{1}{2})$, the lowest W-type is 2_3, and $\mathfrak{z} = C_3$. In terms of Langlands classification, the standard module is induced from the Steinberg representation on A_1, and as a W-representation, $X(4A_1) = \operatorname{Ind}_{W(A_1)}^{W(F_4)}((11))$.

The restrictions of the nearby W-types are:

Nilpotent	$4A_1$	$(A_3A_1)''$	$A_2 3A_1$	$A_3 2A_1$	$D_4(a_1)A_1$	$A_3 A_2 A_1$
W-type	2_3	4_2	1_3	8_3	9_1	4_4
Multiplicity	1	1	1	4	3	3
$C_1 \subset C_2$	0×2	1×1	0×2	$2(0 \times 2)$	0×2	$2(0 \times 2)$
				$2(1 \times 1)$	$2(1 \times 1)$	1×1
$A_1 \subset A_2$	(21)	(21)	(1^3)	$3(21), (1^3)$	$3(21)$	$(21), 2(1^3)$
$\widetilde{A}_1$						
$W(\mathfrak{z})$	3×0	0×3	3×0	$0 \times 3 + 2 \times 1$	1×2	2×1

We need one calculation in the Hecke algebra $\mathbb{H}(C_2, 2, 1)$ of type C_2, $2\Leftarrow 1$. The standard module is induced from the Steinberg representation on C_1, and it has infinitesimal character $(\nu, \frac{1}{2})$. The operators are

$$0 \times 2 : 1, \quad 1 \times 1 : \frac{\frac{3}{2} - \nu}{\frac{3}{2} + \nu}, \quad 0 \times 11 : \frac{(\frac{3}{2} - \nu)(\frac{5}{2} - \nu)}{(\frac{3}{2} + \nu)(\frac{5}{2} + \nu)}.$$

The matching of intertwining operators is with the spherical operators for $\mathbb{H}(C_3, 2, 3)$, the Hecke algebra of type C_3, 2–$2\Leftarrow 3$. The only W-types which give matchings are

    2_3 with 3×0;

    4_2 with 0×3;

    9_1 with 1×2;

    8_1 with 0×12.

The last one is irrelevant for the calculation. To get an inclusion of $\mathcal{U}(4A_1)$ into $\mathcal{SU}(\mathbb{H}(C_3, 2, 3))$, we would have needed to find matchings for the $W(C_3)$-types 2×1 and 12×0 as well. The $W(F_4)$-type 8_3 fails to match 2×1 because of its restriction to $A_1 \subset A_2$, but we will have to use it in the calculation nevertheless. We will also use the operators on 1_3 and 4_1.

The hyperplanes of reducibility for $X(4A_1), (\nu_1, \nu_2, \nu_3))$ are: $\nu_i \pm \nu_j = 2$, $\nu_i = \frac{3}{2}$ (as in $\mathbb{H}(C_3, 2, 3)$), but also $\nu_i = \frac{5}{2}$, and $\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$. Especially the second extra family of hyperplanes of reducibility will affect the unitarity in an essential way.

The Hecke algebra $\mathbb{H}(C_3, 2, 3)$ arises geometrically from a cuspidal local system on $SL(3)^3$ in $Spin(13)$; its dual is parametrized by a subset of the nilpotent orbits in type B_6.. We organize our analysis of the unitary set $\mathcal{U}(4A_1)$ by infinitesimal characters corresponding to these orbits.

$(2^6 1)$: In $\mathbb{H}(C_3, 2, 3)$, these are parameters (ν_1, ν_2, ν_3) in the complement of the hyperplanes $\nu_i \pm \nu_j = 2$, $\nu_i = \frac{3}{2}$. In $\mathbb{H}(C_3, 2, 3)$ the unitary regions are:

  (i) $\{\nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}$.

  (ii) $\{\nu_1 < \frac{3}{2}, \nu_1 + \nu_3 > 2, \nu_2 + \nu_3 < 2\}$.

But the operators on the matched $W(C_3)$-types 3×0, 0×3, 1×2 (equivalently, on the $W(F_4)$-operators 2_3, 4_2, respectively 9_1) are also positive in the region

  (iii) $\{\nu_3 < \frac{3}{2}, \nu_2 > \frac{3}{2}, \nu_1 + \nu_3 > 2, \nu_1 - \nu_3 < 2, \nu_2 + \nu_3 < 2\}$.

In F_4, $X(4A_1)$ becomes reducible also on the hyperplanes $\nu_1 \pm \nu_2 \pm \nu_3 = \frac{3}{2}$, some of which cut the regions (i)–(iii). More specifically, region (i) is cut by the hyperplanes $\nu_1 + \nu_2 + \nu_3 = \frac{3}{2}$ and $\nu_1 + \nu_2 - \nu_3 = \frac{3}{2}$, and region (ii) is cut by $\nu_1 + \nu_2 - \nu_3 = \frac{3}{2}$. We use the operators on 1_2 and 8_4 and these are indefinite in all the resulting (sub)regions except

$$\{\nu_1 + \nu_2 + \nu_3 \le \frac{3}{2}\} \cup \{\nu_1 + \nu_2 - \nu_3 \ge \frac{3}{2}, \nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}. \qquad (5.3.6)$$

In these two remaining regions, one can deform ν_3 to 0, and $X(4A_1)$ stays irreducible, and for $\nu_3 = 0$, it is unitarily induced and unitary. So the parameters in (5.3.6) are unitary. On the hyperplanes $\nu_1 + \nu_2 \pm \nu_3 = \frac{3}{2}$, in (5.3.6), the decomposition is

$$X(4A_1) = \overline{X}(4A_1) + X(A_2 + 3A_1),$$

and the factor $\overline{X}(4A_1)$ is self IM-dual.

(52^4) : The parameter is $(\nu_1, \nu_2, \frac{3}{2})$. In $\mathbb{H}(C_3, 2, 3)$, the corresponding parameters are unitary for $\{\nu_1 < \frac{1}{2}\} \cup \{\nu_2 > \frac{1}{2}, \nu_1 + \nu < 2\}$. However, out of the matched operators, the only nontrivial one is on 9_1, and it is $\frac{(\frac{7}{2} - \nu_1)(\frac{7}{2} - \nu_2)}{(\frac{7}{2} + \nu_1)(\frac{7}{2} + \nu_2)}$. We also need the operators on 1_3 and 4_1:

$$1_3 : \begin{cases} \frac{(3 - (\nu_1 - \nu_2))(3 - (\nu_1 + \nu_2))}{(3 + (\nu_1 - \nu_2))(3 + (\nu_1 + \nu_2))}, & \nu_1 \neq \nu_2 \\ 0, & \nu_1 = \nu_2 \end{cases}, \quad 4_1 : \frac{(\frac{5}{2} - \nu_1)(\frac{5}{2} - \nu_2)(\frac{7}{2} - \nu_1)(\frac{7}{2} - \nu_2)}{(\frac{5}{2} + \nu_1)(\frac{5}{2} + \nu_2)(\frac{7}{2} + \nu_1)(\frac{7}{2} + \nu_2)}.$$

From these calculations, it follows that for $\nu_1 \neq \nu_2$, the only unitary parameters are in the region $\{\nu_1 + \nu_2 < 3, \nu_1 < \frac{5}{2}\}$ (and its closure). In this region, the decomposition of the standard module is

$$X(4A_1, (\nu_1, \nu_2, \frac{3}{2})) = \overline{X}(4A_1, (\nu_1, \nu_2, \frac{3}{2})) + \overline{X}((A_3 A_1)'', (\nu_1 + \nu_2, \nu_1 - \nu_2, 1))$$
$$+ X(D_4(a_1)A_1, (\nu_1, \nu_2)).$$

Moreover, $\overline{X}(4A_1, (\nu_1, \nu_2, \frac{3}{2}))$ is the IM-dual of $X(D_4(a_1) + A_1)$, so it is indeed unitary.

When $\nu_1 = \nu_2 = \nu$, the decomposition is

$$X(4A_1, (\nu, \nu, \frac{3}{2})) = \overline{X}(4A_1, (\nu, \nu, \frac{3}{2})) + \overline{X}((A_3 A_1)'', (2\nu, 1, 0))$$
$$\overline{X}(A_2 + 3A_1, (1, -\frac{3}{2} + \nu)) + X(D_4(a_1)A_1, (\nu, \nu)).$$

The factor $\overline{X}(4A_1, (\nu, \nu, \frac{3}{2}))$ is the IM-dual of $X(D_4(a_1)A_1, (\nu, \nu), phi_1)$, so it is unitary if and only if $0 \leq \nu \leq \frac{3}{2}$.

(44221) : The parameters are $(\nu_1, 1 + \nu_2, -1 + \nu_2)$. In $\mathbb{H}(C_3, 2, 3)$, they are unitary in the region $\{\nu_1 < \frac{3}{2}, \nu_2 < \frac{1}{2}\}$. In F_4, the unitary set will be different. Generically, here the standard module decomposes as

$$X(4A_1, (\nu_1, 1 + \nu_2, -1 + \nu_2)) = \overline{X}(4A_1, (\nu_1, 1 + \nu_2, -1 + \nu_2)) + X(A_3 + 2A_1, (\nu_1, \nu_2)),$$

and $\overline{X}(4A_1)$ is the IM-dual of $\overline{X}((A_3 + A_1)'', (\frac{1}{2} + \nu_1, -\frac{1}{2} + \nu_1, 2\nu_2))$, which is unitary for $\{0 \leq \nu_1 < \frac{1}{2}, 0 \leq \nu_2 < \frac{1}{2}\}$.

Similar calculations as in the previous case $((52^4))$ show that the parameters $(\nu_1, 1 + \nu_2, -1 + \nu_2)$ can be unitary only for $\{0 \leq \nu_1 < \frac{1}{2}, 0 \leq \nu_2 < \frac{1}{2}\}$ (by the remark in the previous paragraph, $\overline{X}(4A_1, (\nu_1, 1 + \nu_2, -1 + \nu_2))$ has to be unitary in this region), and on the segment $\nu_1 + 2\nu_2 = \frac{3}{2}$, for $\frac{1}{2} \leq \nu_1 \leq \frac{3}{2}$. For this, we use, in addition to the matched W-types, the operators on $1_3, 4_1, 8_3$.

It remains to analyze the segment $\nu_1 + 2\nu_2 = \frac{3}{2}$. We rewrite the parameters as $\nu_1 = \frac{1}{2} + 2\nu, \nu_2 = \frac{1}{2} - \nu$. Then the factor $\overline{X}(4A_1, (\frac{1}{2} + 2\nu, \frac{1}{2} + \nu, -\frac{3}{2} + \nu))$ is the IM-dual of $\overline{X}(A_3 + 2A_1, (-\frac{1}{2} + \nu, \frac{1}{2} + \nu))$, which is unitary for $0 \leq \nu < \frac{1}{2}$, which is precisely the segment we were looking at. Therefore, this segment is also unitary for $\overline{X}(4A_1)$.

(53221) : The infinitesimal character is $(\nu, \frac{3}{2}, \frac{1}{2})$. In $\mathbb{H}(C_3, 2, 3)$, the corresponding spherical module is unitary for $0 \leq \nu < \frac{3}{2}$.

In F_4, the only nonzero matched operator is on 9_1, and it is $\frac{\frac{7}{2}-\nu}{\frac{7}{2}+\nu}$. The operator on 1_3 is $\frac{(\frac{5}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{7}{2}+\nu)}$. So it remains to check the segment $0 \leq \nu \leq \frac{5}{2}$. For $0 \leq \nu \leq \frac{5}{2}$, $\overline{X}(4A_1, (\nu, \frac{3}{2}, \frac{1}{2}))$ is the IM-dual of $X(D_4(a_1)A_1, (\nu, \frac{1}{2}))$, so it is indeed unitary.

(544) : The infinitesimal character is $(1 + \nu, -1 + \nu, \frac{3}{2})$. In $\mathbb{H}(C_3, 2, 3)$, the corresponding spherical module is unitary for $0 \leq \nu < \frac{3}{2}$. In F_4, the matched operator on 9_4 gives $\frac{(\frac{5}{2}-\nu)(\frac{9}{2}-\nu)}{(\frac{5}{2}+\nu)(\frac{9}{2}+\nu)}$. The operator on 1_2 is $\frac{\frac{3}{2}-\nu}{\frac{3}{2}+\nu}$, which implies the only unitary parameters can be in $0 \leq \nu \leq \frac{3}{2}$. For $0 \leq \nu < \frac{3}{2}$, $\overline{X}(4A_1, (1 + \nu, -1 + \nu, \frac{3}{2}))$ is the IM-dual of $X(D_4(a_1)A_1, (1 + \nu, -1 + \nu))$, so it is unitary.

(661) : The infinitesimal character is $(2 + \nu, \nu, -2 + \nu)$. In $\mathbb{H}(C_3, 2, 3)$, the corresponding spherical module is unitary for $0 \leq \nu < \frac{1}{2}$. In F_4, the matched operators give $4_5 : \frac{(\frac{1}{2}+\nu)(\frac{3}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{1}{2}-\nu)(\frac{3}{2}+\nu)(\frac{7}{2}+\nu)}$, respectively $9_4 : \frac{(\frac{3}{2}-\nu)(\frac{7}{2}-\nu)}{(\frac{3}{2}+\nu)(\frac{7}{2}+\nu)}$. For $0 \leq \nu < \frac{1}{2}$, the factor $\overline{X}(4A_1, (2 + \nu, \nu, -2 + \nu))$ is the IM-dual of $X((A_5)'', (1, -\frac{1}{2} + \nu))$, therefore it is unitary.

(751) : The infinitesimal character is $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$, which is unitary in $\mathbb{H}(C_3, 2, 3)$. In F_4, $\overline{X}(4A_1, (\frac{5}{2}, \frac{3}{2}, \frac{1}{2}))$ is unitary as well, being the IM-dual of $\overline{X}(D_6(a_1), \frac{1}{2})$.

(922) : The infinitesimal character is $(\nu, \frac{7}{2}, \frac{3}{2})$, which in $\mathbb{H}(C_3, 2, 3)$ is unitary for $0 \leq \nu < \frac{1}{2}$. In F_4, the matched operators are 0, but the operators on 1_2 and 8_4 are $\frac{(\frac{1}{2}+\nu)(\frac{13}{2}-\nu)}{(\frac{1}{2}-\nu)(\frac{13}{2}+\nu)}$, respectively $\frac{\frac{13}{2}-\nu}{\frac{13}{2}+\nu}$.

For $0 \leq \nu < \frac{1}{2}$, the factor $\overline{X}(4A_1, (\nu, \frac{7}{2}, \frac{3}{2}))$ is the IM-dual of $X(D_6(a_1), \nu)$, and therefore unitary. But also the point $\nu = \frac{13}{2}$ is unitary, since $\overline{X}(4A_1, (\frac{13}{2}, \frac{7}{2}, \frac{3}{2}))$ is the IM-dual of $X(E_7(a_1))$. (This point has no correspondent in $\mathbb{H}(C_3, 2, 3)$.)

(931) : The infinitesimal character is $(\frac{7}{2}, \frac{3}{2}, \frac{1}{2})$, which is unitary in $\mathbb{H}(C_3, 2, 3)$. It is also unitary in F_4, as $\overline{X}(4A_1, (\frac{7}{2}, \frac{3}{2}, \frac{1}{2}))$ is the IM-dual of $\overline{X}(D_6(a_1), \frac{1}{2})$.

(13) : The infinitesimal character is $(\frac{11}{2}, \frac{7}{2}, \frac{3}{2})$. In $\mathbb{H}(C_3, 2, 3)$ the corresponding spherical module is the trivial representation. But $\overline{X}(4A_1, (\frac{11}{2}, \frac{7}{2}, \frac{3}{2}))$ is not unitary as seen from the operators for (922) at $\nu = \frac{11}{2}$.

5.4. **Spherical modules.** The spherical modules are parameterized by the nilpotent orbit $(3A_1)''$. If a spherical module does not contain the sign W-representation, via IM its unitarity was already determined in the previous section. We record those results next.

Table 4: Spherical unitary modules for $\mathbb{H}(F_4, (1, 2))$

Nilpotent	Central character	Unitary
E_7	$(\frac{17}{2}, \frac{9}{2}, \frac{5}{2}, \frac{1}{2})$	
$E_7(a_2)$	$(\frac{11}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	
D_6	$(\nu, \frac{9}{2}, \frac{5}{2}, \frac{1}{2})$	$\{0 \leq \nu \leq \frac{1}{2}\}$
$D_5 + A_1$	$(2 + \nu, -2 + \nu, \frac{5}{2}, \frac{1}{2})$	$\{0 \leq \nu \leq \frac{1}{2}\}$
$E_7(a_5)$	$(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	
$D_6(a_2)$	$(\nu, \frac{5}{2}, \frac{3}{2}, \frac{1}{2})$	$\{0 \leq \nu < \frac{1}{2}\}$
$A_5 + A_1$	$(\frac{1}{4}, \frac{7}{4}, -\frac{1}{4}, -\frac{9}{4}) + \nu(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$\{0 \leq \nu < \frac{1}{2}\}$
$(A_5)''$	$(\nu_2 + \frac{3\nu_1}{2} + 2 + \frac{\nu_1}{2}, \frac{\nu_1}{2}, -2 + \frac{\nu_1}{2})$	$\{3\nu_1 + 2\nu_2 < 1\}$

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Table 4 – continued from previous page

		$\{2\nu_1 + \nu_2 < 1 < 3\nu_1 + \nu_2\}$
$D_4 + A_1$	$(\nu_1, \nu_2, \frac{5}{2}, \frac{1}{2})$	$\{\nu_1 + \nu_2 < 1\}$
		$\{\nu_1 - \nu_2 > 1, \nu_1 < \frac{3}{2}\}$
	$(\frac{1}{2}+\nu, -\frac{1}{2}+\nu, \frac{5}{2}, \frac{1}{2})$	$\{0 \le \nu < \frac{3}{2}\}$
$A_3 A_2 A_1$	$(\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{1}{2}) + \nu(2,1,1,0)$	$\{0 \le \nu < 1\}$
$A_3 + 2A_1$	$(\nu_1, 1+\nu_2, -1+\nu_2, \frac{1}{2})$	$\{0 \le \nu_1 < \frac{1}{2}, \nu_1 + 2\nu_2 < \frac{3}{2}\}$
		$\{0 \le \nu_1 < \frac{1}{2}, 0 \le \nu_2 < \frac{3}{2}, 2\nu_2 - \nu_1 > \frac{3}{2}\}$
	$(1+\nu, -1+\nu, \frac{1}{2}, \frac{1}{2})$	$\{0 \le \nu_1 < \frac{1}{2}\} \cup \{1 < \nu < \frac{3}{2}\}$
$(A_3 + A_1)$"	$(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1-\nu_2}{2}, 1+\frac{\nu_3}{2}, -1+\frac{\nu_3}{2})$	$\{0 \le \nu_3 \le \nu_2 \le \nu_1 < 1 - \nu_2\}$
		$\{0 \le \nu_3 \le \nu_2 < 1 - \nu_3 < \nu_1 < 1\}$
	$(\nu_1, \nu_2, \frac{3}{2}, \frac{1}{2})$	$\{0 \le \nu_2 \le \nu_1 < \frac{1}{2}\}$
$A_2 + 3A_1$	$(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}) + \nu_1(2,1,1,0)$	$\{2\nu_1 + \nu_2 < 1\}$
	$+\nu_2(1,1,0,0)$	
$4A_1$	$(\nu_1, \nu_2, \nu_3, \frac{1}{2})$	$\{\nu_1 + \nu_2 + \nu_3 < \frac{3}{2}\}$
		$\{\nu_1 + \nu_2 - \nu_3 > \frac{3}{2}, \nu_1 < \frac{3}{2}, \nu_1 + \nu_2 < 2\}$
$(3A_1)''$	$(\nu_1, \nu_2, \nu_3, \nu_4)$	regions $\mathcal{F}_1 - \mathcal{F}_5$ in section 5.5

5.5. The 0-complementary series. In this section we determine the unitary irreducible spherical series $X(\nu)$. The parameter $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$ is assumed in the dominant Weyl chamber $\mathcal{C}$. This is partitioned by the hyperplanes

$$\langle \alpha_l, \nu \rangle = 1, \ \alpha_l \text{ long root and } \langle \alpha_s, \nu \rangle = c, \alpha_s \text{ short root.} \qquad (5.5.1)$$

We assume first that $c > 1$ is arbitrary. The principal series $X(\nu)$ is reducible precisely when ν is on one of the hyperplanes in (5.5.1). If $\mathcal{F}$ is an open connected component of the complement of (5.5.1) in $\mathcal{C}$ (we call $\mathcal{F}$ a *region*), then all the intertwining operators $r_\sigma(\nu)$, $\sigma \in \widehat{W}$, are invertible, and therefore have constant signature in $\mathcal{F}$. We say that the region $\mathcal{F}$ is unitary if $X(\nu)$ is unitary for all (equivalently, any) $\nu \in \mathcal{F}$. The walls of any region $\mathcal{F}$ are of the form (5.5.1), or of the form $\langle \alpha, \nu \rangle = 0$, for $\alpha \in \Pi$.

Proposition. *Consider the half-space* $K = \{\nu : \langle \epsilon_1 + \epsilon_2, \nu \rangle < c\}$. *The unitary regions* $\mathcal{F}$ *in* $\mathcal{C} \cap K$ *are:*

($\mathcal{F}_1$) $\{2\nu_1 < 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_2$) $\{\nu_1 + \nu_2 + \nu_3 + \nu_4 > 1, \ \nu_1 + \nu_2 + \nu_3 - \nu_4 < 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_3$) $\{\nu_1 + \nu_2 - \nu_3 + \nu_4 > 1, \ \nu_1 - \nu_2 + \nu_3 + \nu_4 < 1, \ \nu_1 + \nu_2 - \nu_3 - \nu_4 < 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_4$) $\{\nu_1 - \nu_2 + \nu_3 + \nu_4 < 1, \ 2\nu_2 > 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_5$) $\{2\nu_2 > 1, \ \nu_1 - \nu_2 + \nu_3 - \nu_4 > 1, \ 2\nu_3 < 1, \ \nu_1 - \nu_2 - \nu_3 + \nu_4 < 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_6$) $\{2\nu_2 > 1, \ \nu_1 - \nu_2 - \nu_3 - \nu_4 > 1, \ 2\nu_3 < 1, \ \nu_1 + \nu_2 < c\}$;
($\mathcal{F}_7$) $\{\nu_1 - \nu_2 - \nu_3 - \nu_4 > 1, \ 2\nu_4 > 1, \ \nu_1 + \nu_2 < c\}$.

Proof. The proof is a case by case analysis, which we sketch here.

Note that $\epsilon_1 + \epsilon_2$ is the highest short root of F_4. The condition that $\mathcal{F}$ be in the half-space K means that the walls of $\mathcal{F}$ can only be of the form $\langle \alpha_l, \nu \rangle = 1$, $\langle \alpha, \nu \rangle = 0$ (or $\nu_1 + \nu_2 = c$). From the partial order relation among the (long) roots, we see that there are 19 such regions. A case by case analysis gives that each of these regions has a wall of the form $\langle \alpha, \nu \rangle = 0$.

If $\mathcal{F}$ has a wall of the form $\langle \alpha, \nu \rangle = 0$, then on this wall, $X(\nu)$ is unitarily induced irreducibly from a principal series $X_M(\nu')$ for a Hecke subalgebra H_M, with M Levi

of type B_3 or C_3. In [BC2], theorem 3.4 and 3.6, the unitary irreducible spherical principal series for the Hecke algebras of type B_n/C_n with arbitrary unequal parameters are determined, so in particular we know the unitarity of $X_M(\nu')$, and therefore of $X(\nu)$.

$\square$

One of the results of [BC2], when the Hecke algebra $\mathbb{H}$ if of type B_n with arbitrary parameter c, is that any region $\mathcal{F}$ on which the highest short root is greater than c, is necessarily not unitary. The similar statement is false in F_4, as seen in the following example.

Example. *The region* $\mathcal{F}_8 = \{\nu_1 - \nu_2 - \nu_3 - \nu_4 > 1,\ 2\nu_4 > 1,\ \nu_1 + \nu_3 > c,\ \nu_1 + \nu_4 < c\}$ *is unitary.*

Proof. The region $\mathcal{F}_8$ has a wall given by $\langle \epsilon_2 - \epsilon_3, \nu \rangle = 0$. We deform the parameter ν to this wall, *i.e.* $\nu_2 = \nu_3$. The corresponding module $X(\nu)$ is unitarily induced irreducible from the principal series $X_{B_3}(\nu')$, where $\nu' = (\nu'_1, \nu'_2, \nu'_3)$ satisfy $\nu'_1 - \nu'_2 > 1, \nu'_2 - \nu'_3 > 1$, and $\nu'_1 < c$. These parameters ν' are unitary in B_3, *cf.* [BC2], theorem 3.6. $\square$

In order to show that the regions $\mathcal{F}_1 - \mathcal{F}_8$ are the *only* unitary regions, we determined by brute force computer calculations the signatures of $r_\sigma(\nu)$, with $\sigma \in \{1_1, 4_2, 9_1\}$, and $\nu \in \mathcal{F}$, for $\mathcal{F}$ any region in $\mathcal{C}$.

Corollary. *In the case of the geometric Hecke algebra* $\mathbb{H}(F_4, (1,2))$, *the* 0-*complementary series is given by the regions* $\mathcal{F}_1 - \mathcal{F}_5$ *from proposition 5.5.*

(When $c = 2$, the regions $\mathcal{F}_6 - \mathcal{F}_8$, in the notation as before, become empty.)

5.6. W-structure. We record the W-structure of standard modules for $\mathbb{H}(F_4, (1,2))$ as follows from the intertwining operator calculations of previous sections. The correspondence is normalized so that the nilpotent E_7 corresponds to the sign representation 1_4, and $(3A_1)''$ to the trivial 1_1. For induced representations and labeling of irreducible characters of the Weyl group of F_4, see for example [Al].

Table 5: W-structure of standard modules of $\mathbb{H}(F_4, (1,2))$

$\mathcal{O}$	$\mu(\mathcal{O}, \phi)$	W-structure
E_7	1_4	1_4
$E_7(a_1)$	2_4	2_4
$E_7(a_2)$	4_5	$4_5 + 1_4$
$E_7(a_3)$	8_4	$8_4 + 2_4$
$E_7(a_3)$	1_2	1_2
D_6	9_4	$\mathrm{Ind}_{C_3}^{F_4}(0 \times 111)$
$E_7(a_4)$	4_3	$4_3 + 1_2$
	2_2	2_2
$D_6(a_1)$	9_2	$\mathrm{Ind}_{C_3}^{F_4}(111 \times 0)$
$D_5 + A_1$	8_2	$\mathrm{Ind}_{B_3}^{F_4}(0 \times 111)$
$E_7(a_5)$	12_1	$12_1 + 8_4 + 4_5 + 8_2 + 9_4 + 1_4$
	6_2	$6_2 + 4_5$
$D_6(a_2)$	16_1	$\mathrm{Ind}_{C_3}^{F_4}(1 \times 11 + 0 \times 111)$
$A_5 + A_1$	6_1	$\mathrm{Ind}_{A_2 + A_1}^{F_4}((111) \otimes (11))$

Table 5 – continued from previous page

$D_5(a_1) + A_1$	4_1	$\mathrm{Ind}_{B_3}^{F_4}(0 \times 12)$
$(A_5)''$	8_1	$\mathrm{Ind}_{A_2}^{F_4}((111))$
$D_4 + A_1$	9_3	$\mathrm{Ind}_{C_3}^{F_4}(0 \times 11)$
$A_3 A_2 A_1$	4_4	$\mathrm{Ind}_{A_2 + \tilde{A}_1}^{F_4}((111) \otimes (11))$
$D_4(a_1) + A_1$	9_1	$\mathrm{Ind}_{B_3}^{F_4}(1 \times 2 + 0 \times 12)$
	2_1	$\mathrm{Ind}_{B_3}^{F_4}(0 \times 3)$
$A_3 + 2A_1$	8_3	$\mathrm{Ind}_{A_1 + \tilde{A}_1}^{F_4}((11) \otimes (11))$
$(A_3 + A_1)''$	4_2	$\mathrm{Ind}_{\tilde{A}_1}^{F_4}((11))$
$A_2 + 3A_1$	1_3	$\mathrm{Ind}_{A_2}^{F_4}((111))$
$4A_1$	2_3	$\mathrm{Ind}_{A_1}^{F_4}((11))$
$(3A_1)''$	1_1	$\mathrm{Ind}_1^{F_4}((1))$

As mentioned in the introduction, we verified that the dimensions of the discrete series modules above agree with the dimensions obtained from the weight spaces in [Re1]. In [Re1], one uses the realization of this algebra in E_8, and employs the notation of [Spa2] for nilpotent orbits, so to compare the two lists, one needs to add an "A_1" to our notation of nilpotent orbits.

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